A STUDY OF STRESS DISTRIBUTION IN A CYLINDRICALLY ANISOTROPIC BODY SUBJECTED TO ECCENTRIC LOADING

by

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CERTIFICATE

This is to certify that the thesis entitled, 'A Study of Stress Distribution in a Cylindrically Anisotropic Body Subjected to Eccentric Loading' by Mr. Rajendra Kumar Mishra is a record of work carried out under my supervision and has not been submitted elsewhere for a degree.

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January, 1984 R.K. Mishra

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NOMENCLATURE

r, θ, z	cylindrical coordinate system
Z	axis of anisotropy
O'	centre of gravity
Z'	geometric axis
x', y'	principal axis of inertia
x	polar axis from which the angles θ are measured
R_n , θ_n	component of surface forces
R, θ	component of body forces per unit volume in the coordinate direction r, θ
$\bar{U}$	potential of body forces
ξ, η	coordinates of centre of gravity in the system x, y, z where Z coincides with axis of anisotropy.
S, I_1, I_2	area and principal moments of inertia of the cross section
P	axial force
M_r , M_θ	bending moments
$M_{r\theta}$	twisting moment
a_{ij}	elastic components of a body
U_0, V_0, W_0	constants of integration which are function of r and θ only.
G.F.C.	arbitrary constants
U,V,W	displacements accompanied by deformation

$w_1, w_2, w_3,$ u_o, v_o, w_c	rigid displacement without deformation
$\bar{\theta}$	constant, equal to relative angle of twist
β_{ij}	$a_{ij} - \dfrac{a_{i3}\, a_{j3}}{a_{33}}$ $(i,\ j = 1,2,4,5,6)$
c	a_{11}/a_{22}
2b	$(2a_{12} + a_{66})/a_{22}$
g	$\sqrt{\dfrac{c + 1 + 2b}{c-b^2}}$
α	$\sqrt{\dfrac{1}{c}}$
$\beta\pm$	$\sqrt{\dfrac{c+1+2b(mk)^2}{2c} \pm \sqrt{\left[\dfrac{c+1+2b(mk)^2}{2c}\right]^2 - \left[\dfrac{(mk)^2-1}{c}\right]^2}}$
ν	$\sqrt{\dfrac{c+b^2+(c+1)b}{c-b^2}}$
k	$\sqrt{\dfrac{c+1+2b}{c}}$
k'	$\dfrac{a_{11}}{a_{11}\, a_{22}-a_{12}^{2}}$
σ'	$\dfrac{2}{k\ a_{66}} - \dfrac{a_{12}}{a_{11}}$
σ	σ'/α
D'	$h^3k'/12$
R	applied lateral load per unit area of element
$Q_r\ ,\ Q_\theta$	shear force per unit area of element
$w\ ,\ w_1$	deflection of outer and inner portion of plate
δ	$\sqrt{\alpha}$

ABSTRACT

In this work an attempt has been made to fabricate cylindrically woven glass fibre - epoxy composite material discs and to study the stress distribution of the same under eccentric loading. A theoretical analysis has also been carried out to study the problem of stress distribution in cylindrically anisotropic plates for the most general case. A method of preparing cylindrically anisotropic plates is developed and strain gauge technique is used to study the stress distribution with the plate subjected to eccentric loading.

CHAPTER-1

INTRODUCTION

1.1 Literature Survey:

Attempts were made for the first time by Carrier and Ithaca [1] to provide mathematical analysis of the distribution of stress in thin cylindrically anisotropic plates. They solved the problem for various types of loading, obtaining the general solution under arbitrary boundary conditions. Around the same time Lekhnitsky [6] studied extensively the plane problems in theory of elasticity for a body with cylindrical anisotropy. Independent work were also carried out by other investigators [3], [4], [5].

The problem of bending of a cylindrically anisotropic plate was investigated by Carrier et al [2] for central loading and Sen Gupta [7] for eccentric loading. Sen Gupta investigated the problem of the bending of a cylindrically anisotropic circular plate of uniform thickness under eccentric concentrated load for the clamped edge mode. Few other investigators Salzman and Patel [8] Kirmser, Haung and Woo [9], Pandalai and Patel [10],

Minkarah and Hoppmann [11] and Bellini and Dzialo [12] have attempted some of the problems of cylindrically anisotropic bodies subjected to dynamic loading. Although, attempts have been made by many investigators in analysing the problem, during the course of literature survey, it has been observed that very little work has been reported so far as material properties of cylindrically reinforced composite is concerned. It has been observed that the experimental determination of material properties of a rectilinearly reinforced composite is straight forward and only tensile specimen are needed, but such simple specimen cannot be used for the case of cylindrically reinforced composites.

The mechanical properties of composite materials are studied from two approaches viz. micromechanics and macromechanics. In the micromechanics approach the interaction of the constituent materials is examined on a microscopic level, whereas in the macromechanics approach the composite material is presumed homogeneous and the effects of the constituent materials are considered as averaged properties of the composite. Dhoopar et al [13], Paul [14], Hermann and Chan [15], Adams and Tsai [16], and Helpin and Tsai [17], have applied micromechanics approach to study the material properties of composite materials.

In the macromechanics approach the average properties of material are required. These are usually determined experimentally. Sachse [18], Tauchert and Guzelsu [19], Adams and Thomas [20] have investigated the rectilinearly orthotropic composite. However very little work has been carried out to investigate the cylindrically anisotropic specimen. Hermann and Chan [15] have suggested a set of six experiments to be conducted on different cylindrical specimens for characterizing the mechanical properties of cylindrically anisotropic plates.

In a recently reported investigation carried out by P.K. Sinha [21] an attempt has been made in characterization of material properties of cylindrically reinforced anisotropic composite. It is now well established that the material properties of the composite material depends on the fibre volume fraction. For evaluating the material properties, the existing approach is to use the properties of a rectilinearly orthotropic composite of the same composition and constituents. The radial dependence of the properties of cylindrically anisotropic plates has been found to be negligible. However Sinha et al [22] have carried out experimental investigation to study the effect of radius. They have established that within a small core at the centre the material properties are strongly dependent on radius. It was also observed that the material

behaviour is isotropic at the centre, whereas the properties are orthotropic and uniform in the region outside the core.

Carrier and Ithaca [1] have considered an isotropic homogeneous core extending upto 0.1 r of the centre. Sinha et al [22] have shown that a great resemblance exists between experimental and theoretical results, if the isotropic homogeneous core extends only upto 0.01 r, leaving the material properties away from the core region as orthotropic and uniform throughout. Both the investigator have, in their mathematical model taken into account, the material singularity at the centre by assuming an isotropic homogeneous core and have shown that a pin hole at the centre does not affect the stress distribution significantly. The level of stress near the hole remains very low.

In another paper Carrier et al [2] have studied the problems of bending of cylindrically anisotropic circular discs for different boundary conditions. The problem of a cylindrically anisotropic disc subjected to eccentric loading was studied by Sen Gupta [7]. Using a mathematical model he has obtained analytical relations for the moments and the deflection.

1.2 Present Work:

In the present work cylindrically anisotropic disc was subjected to eccentric loading in a simply supported mode. The work has been classified in three groups.

1) General formulation of plane stress problem of a cylindrically anisotropic body.

2) Development of the fibre winding technique for the fabrication of cylindrically reinforced composite discs.

3) Mapping of stress field developed when a concentrated eccentric load is applied.

Most of the investigator have analyzed the problem of cylindrically anisotropic body only for plane stress mode. In the present work an attempt has been made to provide general formulation of the problem of cylindrically anisotropic body subjected to bending moments and axial forces. The cases of plane stress and plane strain can be deduced from this general formulation.

Sinha had developed a hand winding set up for the fabrication of composite material discs. In the present work an automatic set up was made for the fibre winding by converting a lathe machine into a winding

mechanism. In hand winding the rotation of the wheel does not remains uniform throughout giving rise to non-uniform pressure exerted at different portion of the discs. In the automatic process developed for the present work the pressure applied remains uniform which eliminates the squeezing operations, resorted to in the hand winding technique to provide uniformity.

Till now, little effort is made to study the variation of stresses when a cylindrically anisotropic disc has been subjected to eccentric loading. Sinha et al [21] have investigated the stress distribution in a cylindrically anisotropic plate subjected to diametral pressure using photoelastic technique. In the present work the stress distribution in a cylindrically aniso-tropic plate subjected to eccentric loading is considered using strain gauge technique.

CHAPTER-2

GENERAL FORMULATION OF PLANE STRESS PROBLEM OF A CYLINDRICALLY ANISOTROPIC BODY

2.1 Introduction:

The heterogeneous fibrous composites, in general are assumed to be homogeneous and anisotropic material for the purpose of stress analysis by the macromechanics approach. As a particular example, circumferentially reinforced composites can be idealized as cylindrically anisotropic homogeneous materials. Carrier and Ithaca [1] have derived the governing partial differential equation for the plane stress problems without body forces in a similar manner as reported by Timoshenko for an isotropic material. The general solution has been reported by Carrier[1] for the simplified case of a plane stress problem for a cylindrically anisotropic homogeneous material. Here, in this work the governing equation has been formulated for the most general case of the deformation of a rod under the influence of tension, bending and body forces. However, the body possesses cylindrical anisotropy in which the stresses do not vary along the generator.

2.2 General Formulation:

General formulation of a plane stress problem is considered in two parts.

a) Derivation of a governing equation for a body bounded by an arbitrary cylindrical surface and possessing cylindrical anisotropy of the most general kind.

b) Simplification of this most general equation to obtain the governing equation for the plane stress problem for cylindrically anisotropic homogeneous material.

Assumptions -

Let us assume that 1) the axis of anisotropy is parallel to the generator of the cylindrical surface 2) the stresses act on the planes normal to the generator and do not vary along the generator and, 3) the body forces are derived from a potential function.

The generalized Hooke's law is stated as -

$$e_\theta = a_{11}\sigma_\theta + a_{12}\sigma_r + a_{13}\sigma_z + a_{14}\tau_{\theta z} + a_{15}\tau_{rz} + a_{16}\tau_{r\theta}$$

$$e_r = a_{21}\sigma_\theta + a_{22}\sigma_r + a_{23}\sigma_z + a_{24}\tau_{\theta z} + a_{25}\tau_{rz} + a_{26}\tau_{r\theta}$$

Eqn.contd...

$$e_z = a_{31}\sigma_\theta + a_{32}\sigma_r + a_{33}\sigma_z + a_{34}\tau_{\theta z} + a_{35}\tau_{rz} + a_{36}\tau_{r\theta}$$

$$\gamma_{r\theta} = a_{16}\sigma_\theta + a_{26}\sigma_r + a_{36}\sigma_z + a_{46}\tau_{\theta z} + a_{56}\tau_{rz} + a_{66}\tau_{r\theta} \qquad \ldots\ldots \qquad (2.1)$$

a_{ij} are the elastic constants of an anisotropic body - the coefficient of deformation. The strains ε_r, ε_θ etc. are connected with the components of displacement (considered to be small) by the formulae

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\theta = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_r}{r}, \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}, \quad \gamma_{rz} = \frac{\partial w}{\partial r} + \frac{\partial u_r}{\partial z}, \quad \gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \qquad \ldots\ldots \qquad (2.2)$$

Let us introduce the notation

$$B = a_{13}\sigma_\theta + a_{23}\sigma_r + a_{33}\sigma_z + a_{34}\tau_{\theta z} + a_{35}\tau_{rz} + a_{36}\tau_{r\theta}$$

On integrating the third, fourth and sixth equations of the generalized Hooke's law (2.1)

$$\varepsilon_z = \frac{\partial w}{\partial z} = B$$

On integrating with respect to z -

$$w = ZB + W_0 (r, \theta) \qquad (2.3)$$

similarly from equations fourth and sixth of (2.1)

$$\gamma_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta} \; ; \quad \frac{\partial u_\theta}{\partial z} = \gamma_{\theta z} - \frac{1}{r}\left[Z\frac{\partial B}{\partial \theta} + \frac{\partial W_0}{\partial \theta} \right]$$

which on integrating with respect to z after substituting the value of $\gamma_{\theta z}$ gives

$$u_\theta = z\left[a_{14}\sigma_\theta + a_{24}\sigma_r + \ldots\right] - \frac{z^2}{2r}\frac{\partial B}{\partial\theta} - \frac{z}{r}\frac{\partial W_0}{\partial\theta} + V_0[r,\theta] \qquad \ldots \qquad (2.4)$$

similarly from sixth equation of (2.1) we get

$$u_r = -\frac{z^2}{2}\frac{\partial B}{\partial r} + z\left[a_{15}\sigma_\theta + a_{25}\sigma_r + \ldots + a_{56}\tau_{r\theta}\right] - z\frac{\partial W_0}{\partial r} + U_0(r,\theta) \qquad \ldots\ldots \qquad (2.5)$$

The relations of strains ε_r, ε_θ and $\gamma_{r\theta}$ are obtained from first, second and sixth equation of (2.1) as

$$\varepsilon_r = -\frac{z^2}{2}\frac{\partial^2 B}{\partial r^2} - z\frac{\partial^2 W_0}{\partial r^2} + \frac{\partial U_0(r,\theta)}{\partial r} + \frac{\partial}{\partial r} z\left[a_{15}\sigma_\theta + \ldots \ldots + a_{56}\tau_{r\theta}\right] \qquad \ldots \qquad (2.6)$$

$$\varepsilon_\theta = \frac{1}{r} z\frac{\partial}{\partial\theta}\left[a_{14}\sigma_\theta + \ldots + a_{46}\tau_{r\theta}\right] - \frac{z^2}{2r}\frac{\partial^2 B}{\partial\theta^2} - \frac{z}{r^2}\frac{\partial^2 W_0}{\partial\theta^2} + \frac{1}{r}\frac{\partial V_0(r,\theta)}{\partial\theta} - \frac{z^2}{2r}\frac{\partial B}{\partial r} + \frac{z}{r}\left[a_{15}\sigma_\theta + \ldots + a_{56}\tau_{r\theta}\right] - \frac{z}{r}\frac{\partial W_0}{\partial r} + \frac{U_0(r,\theta)}{r} \qquad \ldots\ldots \qquad \ldots \qquad (2.7)$$

$$\gamma_{r\theta} = -\frac{z^2}{r}\frac{\partial^2 B}{\partial r\,\partial\theta} + \frac{\partial}{\partial\theta}\frac{z}{r}\left(a_{15}\sigma_\theta + \ldots + a_{56}\tau_{r\theta}\right) - \frac{2z}{r}\frac{\partial^2 W_0}{\partial r\partial\theta} + \frac{1}{r}\frac{\partial U_0}{\partial\theta} + \frac{\partial}{\partial r} z\left(a_{14}\sigma_\theta + \ldots\right) + \frac{\partial V_0}{\partial r} - \frac{z}{r}\left(a_{14}\sigma_\theta + \ldots\right) + \frac{z^2}{r^2}\frac{\partial B}{\partial\theta} + \frac{2z}{r^2}\frac{\partial W_0}{\partial\theta} - \frac{V_0}{r} \qquad \ldots\ldots \qquad (2.8)$$

The axis of anisotropy is the Z axis of cylindrical coordinate r, θ, z. The polar x axis is taken arbitrarily.

According to assumption (3)

$$P = -\frac{\partial \bar{U}}{\partial r} \quad ; \quad \theta = -\frac{1}{r}\frac{\partial \bar{U}}{\partial \theta} \tag{2.9}$$

The loading of the body is shown in Fig. 2.1. The components of stresses in a body deformed by the load are functions of r and θ only. Hence, strains will not be the functions of Z, thus the generalized Hooke's law yields

$$\frac{\partial^2 B}{\partial r^2} = 0 \ , \ \frac{\partial B}{\partial r} + \frac{1}{r}\frac{\partial^2 B}{\partial \theta^2} = 0 \ , \ \frac{\partial^2}{\partial r \, \partial \theta}\left[\frac{B}{r}\right] = 0 \tag{2.10}$$

$$\frac{\partial}{\partial r}\left[a_{15}\sigma_\theta + \ldots\ldots + a_{56}\tau_{r\theta} - \frac{\partial W_0}{\partial r}\right] = 0$$

$$\frac{\partial}{\partial \theta}\left(a_{15}\sigma_\theta + a_{25}\sigma_r + \ldots + a_{56}\tau_{r\theta} - \frac{\partial W_0}{\partial r}\right) + r\frac{\partial}{\partial r}\left(a_{14}\sigma_\theta + \ldots\right.$$

$$\left.\ldots + a_{46}\tau_{r\theta} - \frac{1}{r}\frac{W_0}{\theta}\right) - \left(a_{14}\sigma_\theta + \ldots + \frac{1}{r}\frac{\partial W_0}{\partial \theta}\right) = 0$$

$$\ldots\ldots \tag{2.11}$$

$$\frac{\partial}{\partial \theta}\left(a_{15}\sigma_\theta + a_{25}\sigma_r + \ldots + a_{56}\tau_{r\theta} - \frac{\partial W_0}{\partial r}\right) + r\frac{\partial}{\partial r}\left(a_{14}\sigma_\theta + \right.$$

$$\left. + a_{24}\sigma_r + \ldots + a_{46}\tau_{r\theta} - \frac{1}{r}\frac{\partial W_0}{\partial \theta}\right) - \left(a_{14}\sigma_\theta + \ldots\right.$$

$$\left. + a_{46}\tau_{r\theta} - \frac{1}{r}\frac{\partial W_0}{\partial \theta}\right) = 0$$

$$\left.\begin{aligned} \frac{\partial U_0}{\partial r} &= a_{11}\sigma_\theta + a_{12}\sigma_r + \ldots\ldots + a_{16}\tau_{r\theta} \\ \frac{1}{r}\frac{\partial V_0}{\partial \theta} + \frac{U_0}{r} &= a_{12}\sigma_\theta + a_{22}\sigma_r + \ldots + a_{26}\tau_{r\theta} \\ \frac{1}{r}\frac{\partial U_0}{\partial \theta} + \frac{\partial V_0}{\partial r} - \frac{V_0}{r} &= a_{16}\sigma_\theta + a_{26}\sigma_r + \ldots + a_{66}\tau_{r\theta} \end{aligned}\right\} \tag{2.12}$$

By integrating

$$B = a_{33}\left[Gr\sin\theta + Fr\cos\theta + C\right] \tag{2.13}$$

and from this

$$\sigma_z = Gr\sin\theta + Fr\cos\theta + C - \frac{1}{a_{33}}\left[a_{13}\sigma_\theta + a_{23}\sigma_r + a_{34}\tau_{\theta z} + a_{35}\tau_{rz} + a_{36}\tau_{r\theta}\right] \quad \ldots \tag{2.14}$$

where as ,

$$\left.\begin{aligned} U_0 &= U + u_0\cos\theta + v_0\sin\theta \\ V_0 &= V - u_0\sin\theta + v_0\cos\theta \\ W_0 &= W + w_1 r\sin\theta - w_2 r\cos\theta + w_0 \end{aligned}\right\} \tag{2.15}$$

Then, from (2.12) and (2.11) we obtain the equations which connect the functions U,V,W with the components of stresses.

$$\frac{\partial U}{\partial r} = \beta_{11}\sigma_\theta + \beta_{12}\sigma_r + \beta_{14}\tau_{rz} + \beta_{15}\tau_{\theta z} + \beta_{16}\tau_{r\theta} + a_{13}\left[Gr\sin\theta + Fr\cos\theta + C\right]$$

$$\frac{1}{r}\frac{\partial V}{\partial\theta} + \frac{U}{r} = \beta_{12}\sigma_\theta + \beta_{22}\sigma_r + \beta_{24}\tau_{rz} + \beta_{25}\tau_{\theta z} + \beta_{26}\tau_{r\theta} + a_{23}\left[Gr\sin\theta + Fr\cos\theta + C\right]$$

$$\frac{1}{r}\frac{\partial U}{\partial\theta} + \frac{\partial V}{\partial r} - \frac{V}{r} = \beta_{16}\sigma_\theta + \beta_{26}\sigma_r + \beta_{46}\tau_{rz} + \beta_{56}\tau_{\theta z} + \beta_{66}\tau_{r\theta} + a_{36}(Gr\sin\theta + Fr\cos\theta + C) \quad \ldots\ldots \tag{2.16}$$

$$\frac{\partial W}{\partial r} = \beta_{15}\sigma_\theta + \beta_{25}\sigma_r + \beta_{45}\tau_{rz} + \beta_{55}\tau_{\theta z} + \beta_{56}\tau_{r\theta} + a_{35}(Gr\sin\theta + Fr\cos\theta + C)$$

Eqn. contd....

$$\frac{1}{r}\frac{\partial \gamma}{\partial \theta} = \beta_{14}\,\sigma_{\theta} + \beta_{24}\,\sigma_{r} - \beta_{44}\,\tau_{rz} + \beta_{45}\,\tau_{\theta z} + \beta_{46}\,\tau_{r\theta}$$
$$+ a_{34}\left[Gr\,\sin\theta + Fr\,\cos\theta + C\right] - \bar{\theta}_{r}. \qquad \ldots\ldots \quad (2.17)$$

On substituting the relationship of B, U_0, V_0, w_0 from (2.13), (2.15) and (2.17) in (2.3), (2.4) and (2.5), we obtain the displacement formulae

$$w = Z\,(Gr\,\sin\theta + Fr\,\cos\theta + C)\,a_{33} + W$$
$$+ w_1\,r\sin\theta - w_2\,r\cos\theta + w_0$$

$$u_{\theta} = -\frac{Z^2}{2}(C\,\cos\theta - F\,\sin\theta)\,a_{33} + V + \theta_{z}r + Z\,(-w_2\,\sin\theta$$
$$- w_1\,\cos\theta) - u_0\,\sin\theta + v_0\,\cos\theta + w_3 r$$

$$u_{r} = \frac{-Z^2}{2}\,(G\,\sin\theta + F\,\cos\theta)\,a_{33} + U + Z\,(w_2\,\cos\theta - w_1\,\sin\theta)$$
$$+ u_0\,\cos\theta + v_0\,\sin\theta \qquad \ldots\ldots \quad (2.18)$$

The equations of equilibrium of a body with cylindrical anisotropy in which the stresses do not depend on Z will have the form

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} - \frac{\partial \bar{U}}{\partial r} = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} - \frac{1}{r}\frac{\partial \bar{U}}{\partial \theta} = 0 \qquad (2.19)$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\tau_{rz}}{r} = 0$$

We can define the stresses as function of stress functions ϕ and Ψ.

$$\sigma_r = \frac{1}{r}\frac{\partial \emptyset}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \emptyset}{\partial \theta^2} + \bar{U} \; ; \; \sigma_\theta = \frac{\partial^2 \emptyset}{\partial r^2} + \bar{U}$$

$$\tau_{r\theta} = - \frac{\partial^2}{\partial r \partial \theta}(\emptyset / r) \; ; \; \tau_{rz} = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} \; ; \; \tau_{\theta z} = - \frac{\partial \Psi}{\partial r}$$

The equilibrium equation, (2.19) will be satisfied by eliminating U and V from. (2.16) and W from (2.17) by means of differentiation; thus we obtain a system of equations satisfied by the stress functions $\emptyset$ and Ψ :

$$D_4\emptyset + D_3\Psi = 2 \left[(a_{13} - a_{23})\, G - a_{36}\, F\right] \frac{\sin\theta}{r}$$

$$+ 2 \left[a_{36}\, G + (a_{13} - a_{23})\, F\right] \frac{\cos\theta}{r} - (\beta_{12} + \beta_{22}) \frac{\partial^2 \bar{U}}{\partial r^2}$$

$$+ (\beta_{16} + \beta_{26}) \frac{1}{r} \frac{\partial^2 \bar{U}}{\partial r \partial \theta} - (\beta_{11} + \beta_{12}) \frac{1}{r^2} \frac{\partial^2 \bar{U}}{\partial \theta^2}$$

$$+ (\beta_{11} - 2\beta_{22} - \beta_{12}) \frac{1}{r} \frac{\partial \bar{U}}{\partial r} + (\beta_{16} + \beta_{26}) \frac{1}{r} \frac{\partial \bar{U}}{\partial \theta}$$

$$D_3'\emptyset + D_2\Psi = (- a_{35}\, G + 2a_{34}\, F) \cos\theta + (2a_{34}\, G + a_{35}\, F) \sin\theta$$

$$+ C\, a_{34} \frac{1}{r} - 2\bar{e} + (\beta_{14} + \beta_{24}) \left(\frac{\partial \bar{U}}{\partial r} - \frac{\bar{U}}{r}\right)$$

$$- (\beta_{15} + \beta_{25}) \frac{1}{r} \frac{\partial \bar{U}}{\partial \theta}$$

..... (2.21)

where D_2 , D_3 , D_3' , D_4 are second, third, third and fourth order differential operators defined as follows

$$D_2 = \beta_{44} \frac{\partial^2}{\partial r^2} - 2\, \beta_{45} \frac{1}{r} \frac{\partial^2}{\partial r \partial \theta} + \beta_{55} \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \beta_{44} \frac{1}{r} \frac{\partial}{\partial r}$$

eqn. contd...

$$D_3 = -\beta_{24}\frac{\partial^3}{\partial r^3} + (\beta_{25}+\beta_{46})\frac{1}{r}\frac{\partial^3}{\partial r^2\partial\theta} - (\beta_{14}+\beta_{56})\frac{1}{r^2}\frac{\partial^3}{\partial r\partial\theta^2}$$

$$+\beta_{15}\frac{1}{r^3}\frac{\partial^3}{\partial\theta^3} + (\beta_{14}-2\beta_{24})\frac{1}{r}\frac{\partial^2}{\partial r^2} + (\beta_{46}-\beta_{15})\frac{1}{r^2}\frac{\partial^2}{\partial r\partial\theta}$$

$$+\beta_{15}\frac{1}{r^3}\frac{\partial}{\partial\theta}$$

$$D'_3 = -\beta_{24}\frac{\partial^3}{\partial r^3} + (\beta_{25}+\beta_{46})\frac{1}{r}\frac{\partial^3}{\partial r^2\partial\theta} - (\beta_{14}+\beta_{56})\frac{1}{r^2}\frac{\partial^3}{\partial r\partial\theta^2}$$

$$+\beta_{15}\frac{1}{r^3}\frac{\partial^3}{\partial\theta^3} - (\beta_{14}+\beta_{24})\frac{1}{r}\frac{\partial^2}{\partial r^2} + (\beta_{15}-\beta_{46})\frac{1}{r^2}\frac{\partial^2}{\partial r\partial\theta}$$

$$+(\beta_{14}+\beta_{56})\frac{1}{r^3}\frac{\partial^2}{\partial\theta^2} + \beta_{46}\frac{1}{r^3}\frac{\partial}{\partial\theta}$$

$$D_4 = \beta_{22}\frac{\partial^4}{\partial r^4} - 2\beta_{26}\frac{1}{r}\frac{\partial^4}{\partial r^3\partial\theta} + (2\beta_{12}+\beta_{66})\frac{1}{r^2}\frac{\partial^4}{\partial r^2\partial\theta^2}$$

$$-2\beta_{16}\frac{1}{r^3}\frac{\partial^4}{\partial r\partial\theta^3} + \beta_{11}\frac{1}{r^4}\frac{\partial^4}{\partial\theta^4} + 2\beta_{22}\frac{1}{r}\frac{\partial^3}{\partial r^3}$$

$$-(2\beta_{12}+\beta_{66})\frac{1}{r^3}\frac{\partial^3}{\partial r\partial\theta^2} + 2\beta_{16}\frac{1}{r^4}\frac{\partial^3}{\partial\theta^3} - \beta_{11}\frac{1}{r^2}\frac{\partial^2}{\partial r^2}$$

$$-2(\beta_{16}+\beta_{26})\frac{1}{r^3}\frac{\partial^2}{\partial r\partial\theta} + (2\beta_{11}+2\beta_{12}+\beta_{66})\frac{1}{r^4}\frac{\partial^2}{\partial\theta^2}$$

$$+\beta_{11}\frac{1}{r^3}\frac{\partial}{\partial r} + 2(\beta_{16}+\beta_{26})\frac{1}{r^4}\frac{\partial}{\partial\theta}$$

...... (2.22)

For a body with a bounded cross section the following conditions must be satisfied.

$$(G\eta + F\xi + C)\,S - \frac{1}{a_{33}}\iint (a_{13}\sigma_\theta + a_{23}\sigma_r + a_{34}\tau_{rz}$$

$$+ a_{35}\tau_{\theta z} + a_{36}\tau_{r\theta})\,dS = P_Z$$

eqn. contd....

$$GI_1 - \frac{1}{a_{33}} \iint (a_{13}\sigma_\theta + a_{23}\sigma_r + \dots + a_{35}\tau_{r\theta})$$

$$(r\sin\theta - \eta)\, dS = M_r$$

$$FI_2 - \frac{1}{a_{33}} \iint (a_{13}\sigma_\theta + a_{23}\sigma_r + \dots + a_{36}\tau_{r\theta})$$

$$(r\cos\theta - \xi)\, dS = M_\theta$$

$$\iint \tau_{\theta z}\, r\, dS = M_{r\theta}$$

$$\dots\dots \qquad (2.23)$$

Where the integrals are taken over the area of cross section. For the given external stresses the condition on the contour of the cross section can be written in the form (n is the direction of external normal):

$$\sigma_r \cos(n,r) + \tau_{r\theta}\cos(n,\theta) = R_n$$

$$\tau_{r\theta}\cos(n,r) + \sigma_\theta \cos(n,\theta) = \Theta_n \qquad (2.24)$$

$$\tau_{rz}\cos(n,r) + \tau_{\theta z}\cos(n,\theta) = C$$

Thus, we obtain the stress functions which satisfy both the system (2.22) and the boundary conditions. These stress functions contain the constants θ, G, F and C.

2.3 Plane Deformation of a Body with Cylindrical Anisotropy

If a body has a plane of elastic symmetry normal to the generator of the cylindrical surface, then, the constants a_{14}, a_{24}, a_{34}, a_{46}, a_{15}, a_{25}, a_{35}, a_{56} are equal to zero and consequently,

$$\beta_{14} = \beta_{24} = \beta_{46} = \beta_{15} = \beta_{25} = \beta_{56} = 0 \qquad (2.25)$$

Thus the formulation involves only the function $\emptyset$.

2.4 Deformation of a Rod Under the Action of an Axial Force and Bending Moments

If the body is acted upon by the stresses which reduce to the axial force P_z and to the bending moments M_r and M_θ act on the ends then the function $\emptyset$ satisfies

$$D_4\emptyset = 2\left[(a_{13} - a_{23})\,G - a_{36}\,F\right]\frac{\sin\theta}{r} + 2\left[a_{36}\,G + (a_{13} - a_{23})\,F\right]\frac{\cos\theta}{r}$$

$$- (\beta_{12}+\beta_{22})\frac{\partial^2\bar{U}}{\partial r^2} + (\beta_{16}+\beta_{26})\frac{1}{r}\frac{\partial^2\bar{U}}{\partial r\partial\theta}$$

$$- (\beta_{11}+\beta_{12})\frac{1}{r^2}\frac{\partial^2\bar{U}}{\partial\theta^2} + (\beta_{11} - 2\beta_{22} - \beta_{12})\frac{1}{r}\frac{\partial\bar{U}}{\partial r}$$

$$+ (\beta_{16}+\beta_{26})\frac{1}{r^2}\frac{\partial\bar{U}}{\partial\theta} \qquad \ldots\ldots \qquad (2.26)$$

The components of stresses are given by

$$\sigma_r = \frac{1}{r}\frac{\partial\emptyset}{\partial r} + \frac{1}{r^2}\frac{\partial^2\emptyset}{\partial\theta^2} + \bar{U}$$

$$\sigma_\theta = \frac{\partial^2\emptyset}{\partial r^2} + \bar{U} \qquad (2.27)$$

$$\tau_{r\theta} = -\frac{\partial^2}{\partial r\partial\theta}(\emptyset/r)$$

$$\tau_{rz} = \tau_{\theta z} = 0$$

$$\sigma_z = Gr\sin\theta + Fr\cos\theta + C - \frac{1}{a_{33}}\left[a_{13}\sigma_r + a_{23}\sigma_\theta + a_{36}\tau_{r\theta}\right]$$

The components of displacements are given by

$$u_r = -\frac{Z^2}{2}(G \sin\theta + F \cos\theta)\, a_{33} + U + Z\,(w_2 \cos\theta - w_1 \sin\theta)$$

$$+ u_0 \cos\theta + v_0 \sin\theta$$

$$u_\theta = -\frac{Z^2}{2}(G \sin\theta + F \cos\theta)\, a_{33} + V + w_3\, r + Z\,(- w_2 \sin\theta$$

$$- w_1 \cos\theta) - u_0 \sin\theta + v_0 \cos\theta$$

$$w = Z\,(Gr \sin\theta + Fr \cos\theta + C) + W + w_1 r \sin\theta - w_2\, r \cos\theta$$

$$+ w_0 \qquad \ldots\ldots \qquad (2.28)$$

2.5 Bending of Cylindrically Anisotropic Disc Under Eccentric Loading

This is a special case of bending of a rod possessing cylindrical anisotropy in which the stresses do not vary along the generator and in which case

$$a_{36} = 0 \quad , \quad a_{13} = a_{23} \qquad (2.29)$$

and body forces are absent.

Thus, the equation which determines the stress function $\emptyset$ becomes homogeneous.

$$D_4\emptyset = 0 \qquad (2.30)$$

where D_4 is defined for this special case as

$$D_4 = \beta_{22}\frac{\partial^4}{\partial r^4} + (2\beta_{12} + \beta_{66})\frac{1}{r^2}\frac{\partial^4}{\partial r^2 \partial\theta^2} + \beta_{11}\frac{1}{r^4}\frac{\partial^4}{\partial\theta^4}$$

$$+ 2\beta_{22}\frac{1}{r}\frac{\partial^3}{\partial r^3} - (2\beta_{12} + \beta_{66})\frac{1}{r^3}\frac{\partial^3}{\partial r\,\partial\theta^2}$$

eqn. contd....

$$- \beta_{11} \frac{1}{r^2} \frac{\partial^2}{\partial r^2} + (2\beta_{11} + 2\beta_{12} + \beta_{66}) \frac{1}{r^4} \frac{\partial^2}{\partial \theta^2}$$

$$+ \beta_{11} \frac{1}{r^3} \frac{\partial}{\partial r} \qquad \ldots\ldots \qquad (2.31)$$

and β_{ij} are reduced to

$$\beta_{11} = a_{11} - \frac{a_{13}^2}{a_{33}}, \quad \beta_{22} = a_{22} - \frac{a_{13}\, a_{23}}{a_{33}}$$

$$\beta_{12} = a_{12} - \frac{a_{13}\, a_{23}}{a_{33}}, \quad \beta_{66} = a_{66} \qquad (2.32)$$

for a case in which thickness is very small compared to the radius of the plate the stress along the thickness, $\sigma_z = 0$. This pertains to plane stress problem, hence $\beta_{11} = a_{11}$, $\beta_{12} = a_{12}$, $\beta_{22} = a_{22}$ and $\beta_{66} = a_{66}$ and the homogeneous equation $D_4\emptyset = 0$ reduces to the same form as obtained by Carrier and Ithaca [1].

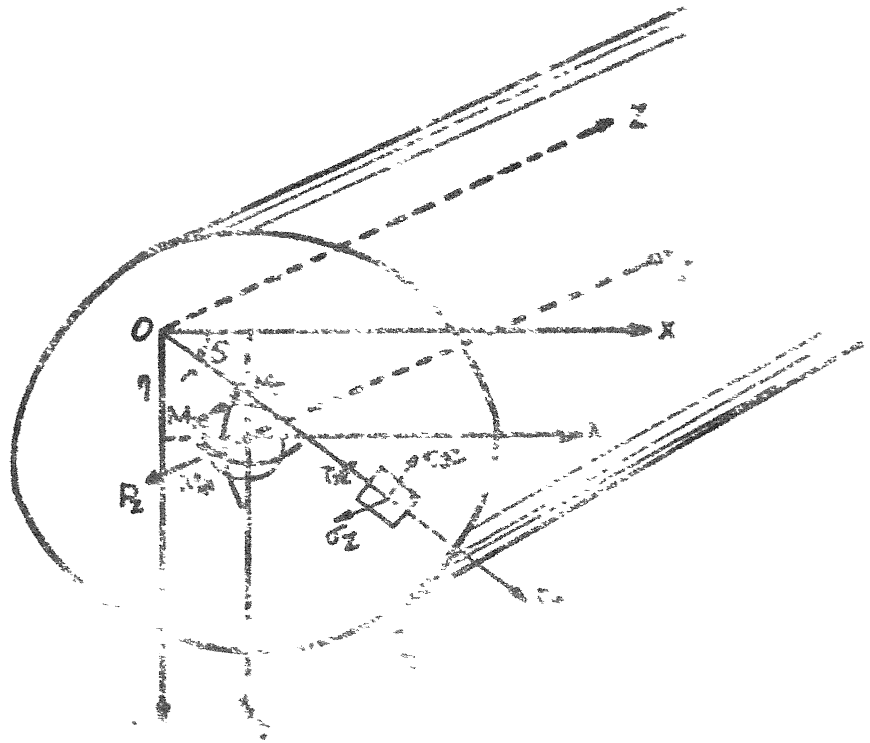
The solution of this equation is obtained by applying separation of variables technique, assuming, $t = \ln r$ and introducing a new function, $\Psi(t,\theta) = e^{-t}\emptyset(r,\theta)$. The equation then have the useful solution of the type

$$\Psi = \Theta(\theta)\, T(t) \qquad \ldots \qquad (2.33)$$

These give the following solutions for $\emptyset$, when the relation between Ψ and $\emptyset$ is again used.

$$
\begin{aligned}
\emptyset = {} & a_0 r^2 + a_1 r^{1+\alpha} + a_2 r^{1-\alpha} + b_0 \theta + b_1 r^2 \theta + c_0 r\, \theta \sin\theta \\
& + c_1 r\, \theta \cos\theta + c_2 r^{1+g}\, \theta \sin\theta + c_3 r^{1+g}\, \theta \cos\theta \\
& + (A r^{1+k} + B r + C r^{1-k} + D r \ln r) \sin\theta \qquad (2.34) \\
& + (E r^{1+k} + F r + G r^{1-k} + H r \ln r) \cos\theta \\
& + \sum_{m=1, mk \neq 1}^{\infty} (A_m\, r^{1+\beta^+} + B_m\, r^{1-\beta^+} + C_m\, r^{1+\beta^-} + D_m\, r^{1-\beta^-}) \sin mk\theta \\
& + \sum_{m=1, mk \neq 1}^{\infty} (E_m\, r^{1+\beta^+} + F_m\, r^{1-\beta^+} + G_m\, r^{1+\beta^-} + H_m\, r^{1-\beta^-}) \cos mk\theta
\end{aligned}
$$

Once the stress function $\emptyset$ is known the components of stresses can be obtained by substituting the expression for $\emptyset$ in equation (2.27).



ic Body Subjected to Moments and Axial Force

CHAPTER-3

BENDING OF CYLINDRICALLY ANISOTROPIC PLATE SUBJECTED TO ECCENTRIC LOADING

3.1 Small Deflection Theory

Small deflection theory approach adopted is similar to the one given by Timoshenko for isotropic case [23]. Taking a sectorial element under general loading the moments and shear forces developed are as shown in Fig. 3.1. The equation of equilibrium derived from a consideration of the couples and forces acting on the sectorial element shown in Fig. 3.1 are same as in isotropic case.

$$\frac{\partial M_r}{\partial r} - \frac{1}{r}\,\frac{\partial M_{r\theta}}{\partial \theta} + \frac{1}{r}\,(M_r - M_\theta) - Q_r = 0$$

$$\frac{1}{r}\,\frac{\partial M_\theta}{\partial \theta} - \frac{\partial M_{r\theta}}{\partial r} - \frac{2}{r}\,M_{r\theta} - Q_\theta = 0 \qquad (3.1)$$

$$\frac{\partial}{\partial r}\,(r\,Q_r) + \frac{\partial Q_\theta}{\partial \theta} + rp = 0$$

where,

$$M_r = -\,D'\left[\frac{d^2w}{dr^2} - \frac{a_{12}}{a_{11}}\left\{\frac{1}{r}\,\frac{dw}{dr} + \frac{1}{r^2}\,\frac{d^2w}{d\theta^2}\right\}\right]$$

$$M_\theta = -\,D'\left[\frac{1}{C}\left(\frac{1}{r}\,\frac{dw}{dr} + \frac{1}{r^2}\,\frac{d^2w}{d\theta^2}\right) - \frac{a_{12}}{a_{11}}\,\frac{d^2w}{dr^2}\right] \qquad (3.2)$$

$$M_{r\theta} = -\,M_{\theta r} = D'\left(\frac{2}{k\,a_{66}}\right)\frac{d}{dr}\left(\frac{1}{r}\,\frac{dw}{d\theta}\right)$$

and the differential equation becomes

$$\left[\frac{1}{r^4}\frac{d^4}{d\theta^4} + \frac{2\sigma}{r^2}\frac{d^4}{dr^2\, d\theta^2} - \frac{2\sigma}{r^3}\frac{d^3}{dr\, d\theta^2} + \frac{2(1+\sigma)}{r^4}\frac{d^2}{d\theta^2} + C\frac{d^4}{dr^4} + \frac{2C}{r}\frac{d^3}{dr^3} - \frac{1}{r^2}\frac{d^2}{dr^2} + \frac{1}{r^3}\frac{d}{dr}\right] w = \frac{RC}{D'} \qquad (3.3)$$

To study the bending of cylindrically anisotropic circular plates under a concentrated eccentric load P , it is assumed that the load P is applied at a point A at a distance R from the centre O of the plate. Considering the plate as divided into two parts by the cylindrical section of radius R, the deflection for each of these portions of the plate can be found from the solution of the equation -

$$\left[\frac{1}{r^4}\frac{d^4}{d\theta^4} + \frac{2\sigma}{r^2}\frac{d^4}{dr^2\, d\theta^2} - \frac{2\sigma}{r^3}\frac{d^3}{dr\, d\theta^2} + \frac{2(\sigma+1)}{r^4}\frac{d^2}{d\theta^2} + C\frac{d^4}{dr^4} + \frac{2C}{r}\frac{d^3}{dr^3} - \frac{1}{r^2}\frac{d^2}{dr^2} + \frac{1}{r^3}\frac{d}{dr}\right] w = 0 \qquad (3.4)$$

Assuming the solution of this equation to be -

$$w = R_0 + \sum_{m=1}^{\infty} R_m \cos m\theta \quad \ldots \qquad (3.5)$$

where,

$$R_0 = A_0\, r^{1+\delta} + B_0 + C_0\, r^2 + D_0\, r^{1-\delta}$$

$$R_1 = A_1\, r^{1+k'} + B_1\, r + C_1\, r^{1-k'} + D_1\, r \log_e r$$

and,

$$R_m = A_m\, r^{1+p_m+q_m} + B_m\, r^{1+p_m-q_m} + C_m\, r^{1-p_m+q_m} + D_m\, r^{1-p_m-q_m}.$$

Similarly the expressions for the functions R_0', R_1', ... and R_m', corresponding to the inner portion of the plate are gotten. The constants for inner portion are A_m', B_m', C_m', D_m' instead of A_m, B_m, C_m and D_m and from the condition that moments, deflection and shear forces must be finite at the centre of the plate, we have

$$
\begin{aligned}
&C_0' = D_0' = 0 \\
&C_1' = D_1' = 0 \qquad \ldots \qquad (3.7)\\
&\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots \\
&C_m' = D_m' = 0
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&R_0' = A_0' \, r^{1+\delta} + B_0' \\
&R_1' = A_1' \, r^{1+k'} + B_1' \, r \qquad (3.8)\\
&R_m' = A_m \, r^{1+p_m+q_m} + B_m' \, r^{1+p_m-q_m}
\end{aligned}
$$

Thus, for each term of series six terms have to be obtained - four for the outer portion of the plate and two for the inner portion.

These can be obtained from the boundary conditions and the conditions of continuity along the circle of radius R.

The required equations for simply supported end condition are

$$(w)_{r=a} = 0$$

$$\left(\frac{d^2w}{dr^2}\right)_{r=a} = 0 \tag{3.9}$$

From the condition of continuity at circle of radius R

$$w = w_1 \; , \; \frac{dw}{dr} = \frac{dw_1}{dr}, \; \frac{d^2w}{dr^2} = \frac{d^2w_1}{dr^2}\Bigg|_{r=R} \tag{3.10}$$

and the last condition is obtained from the condition of continuity of shear force at all points of circle at r=R except at the point A. Representing the load in a series form

$$\frac{P}{\pi R}\left\{\frac{1}{2} + \sum_{m=1}^{\infty} \cos m\theta\right\}$$

the expression for shear force can be obtained from (3.1) and (3.2)

$$Q_r = - D'\left[\frac{d^3}{dr^3} + \frac{1}{r}\frac{d^2}{dr^2} - \frac{\alpha}{r^2}\frac{d}{dr} + \frac{\sigma'}{r^2}\frac{d^3}{dr\,d\theta^2} - \frac{\alpha+\sigma'}{r^3}\frac{d^2}{d\theta^2}\right]w$$

The last condition is

$$D'\left[\frac{d^3}{dr^3} + \frac{1}{r}\frac{d^2}{dr^2} - \frac{\alpha}{r^2}\frac{d}{dr} + \frac{\sigma'}{r^2}\frac{d^3}{dr\,d\theta^2} - \frac{\alpha+\sigma'}{r^3}\frac{d^2}{d\theta^2}\right]_{r=R} w$$

$$- D'\left[\frac{d^3}{dr^3} + \frac{1}{r}\frac{d^2}{dr^2} - \frac{\alpha}{r^2}\frac{d}{dr} + \frac{\sigma'}{r^2}\frac{d^3}{dr\,d\theta^2}\right.$$

$$\left.- \frac{\alpha+\sigma'}{r^3}\frac{d^2}{d\theta^2}\right]_{r=R} w_1 = \frac{P}{\pi R}\left\{\frac{1}{2} + \sum_{m=1}^{\infty} \cos m\theta\right\} \tag{3.11}$$

With the help of equations (3.9), (3.10) and (3.11) the functions R_m and R_m' can be obtained. Since R_m diminishes rapidly as m increases, only a few terms of the series Eq. (3.4) need be calculated to obtain the deflection at any point with sufficient accuracy.

The solution R_0 which is independent of the angle θ represents symmetrical bending of circular plates.

3.2 Evaluation of Deflection When Central Concentrated Load is Applied

The evaluation of deflection for the eccentric load is very cumbersome by this analytical method. However Sengupta [7] and Carrier et al [2] have obtained deflection for a circular plate under concentrated load with clamped edge as

$$w = - \frac{Pa^2}{4\,\pi\,(\alpha-1)\,D'}\left[(r/a)^2 - \frac{2}{(1+\sqrt{\alpha})}\,(r/a)^{(1+\sqrt{\alpha})} - \frac{(\sqrt{\alpha}-1)}{(\sqrt{\alpha}+1)}\right] \tag{3.12}$$

where as Timoshenko [23] has obtained the expression for deflection for an isotropic circular plate under concentrated load for fixed ends as

$$w = \frac{Pr^2}{8\,\pi\,D'}\,\log\frac{r}{a} + \frac{P}{16\,\pi\,D'}\,(a^2 - r^2) \tag{3.13}$$

and for simply supported edge condition by

$$w = \frac{P}{16 \pi D'} \left[\frac{3+\nu}{1+\nu} (a^2 - r^2) + 2r^2 \log \frac{r}{a} \right] \qquad (3.14)$$

where ν is the poison's ratio.

3.3 Comparison of Radial Strain for Central Loading

The relation between strain and displacement are

$$\varepsilon_r = \frac{\partial u}{\partial r}$$

$$\varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad (3.15)$$

where

$$u = - Z \left(\partial w / \partial r \right)$$

$$v = - Z/r \left(\partial w / \partial \theta \right) \qquad (3.16)$$

The expression of deflection for the isotropic and cylindrically anisotropic cases for the clamped edge conditions are given by (3.13) and (3.12) and for simply supported end condition by (3.14).

For a central load of 10 kg. and taking the appropriate values of D' the radial strains for various modes thus developed for the plate are as given in Table 3.1.

Table 3.1

	Particular	r (cm)						
		0	1	3	4	6	7.5	9
ε_r $\frac{\mu m}{m}$	Experimental	-	-	96	48	44	18	-30
	(Analytical) Clamped edge Isotropic plate	∞	150	36	6	-36	-60	-76
	Simply supported Isotropic plate	∞	227	115	84	41	18	-5
	Clamped edge anisotropic	654	466	115	77	54	-13.9	-87

Fig. 3.2 gives the comparison of experimentally obtained strains with theoretical values.

where

$D' = 1304.25$ kgf-cm for anisotropic material

$= 326.06$ kgf-cm for isotropic material

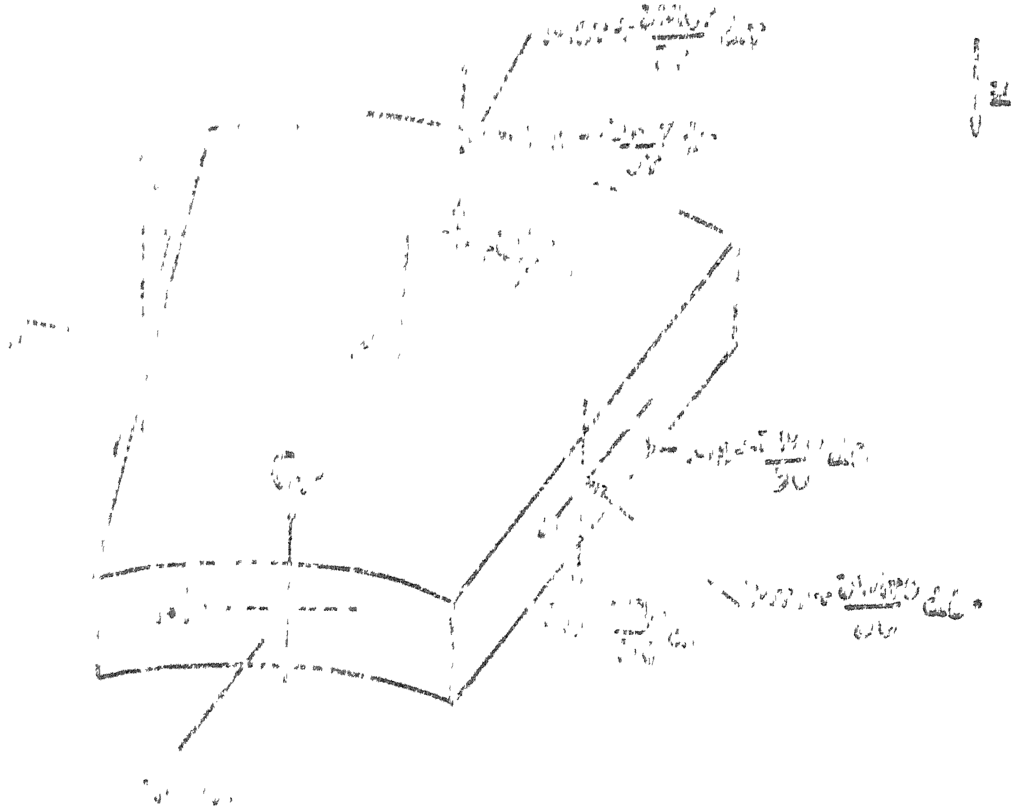
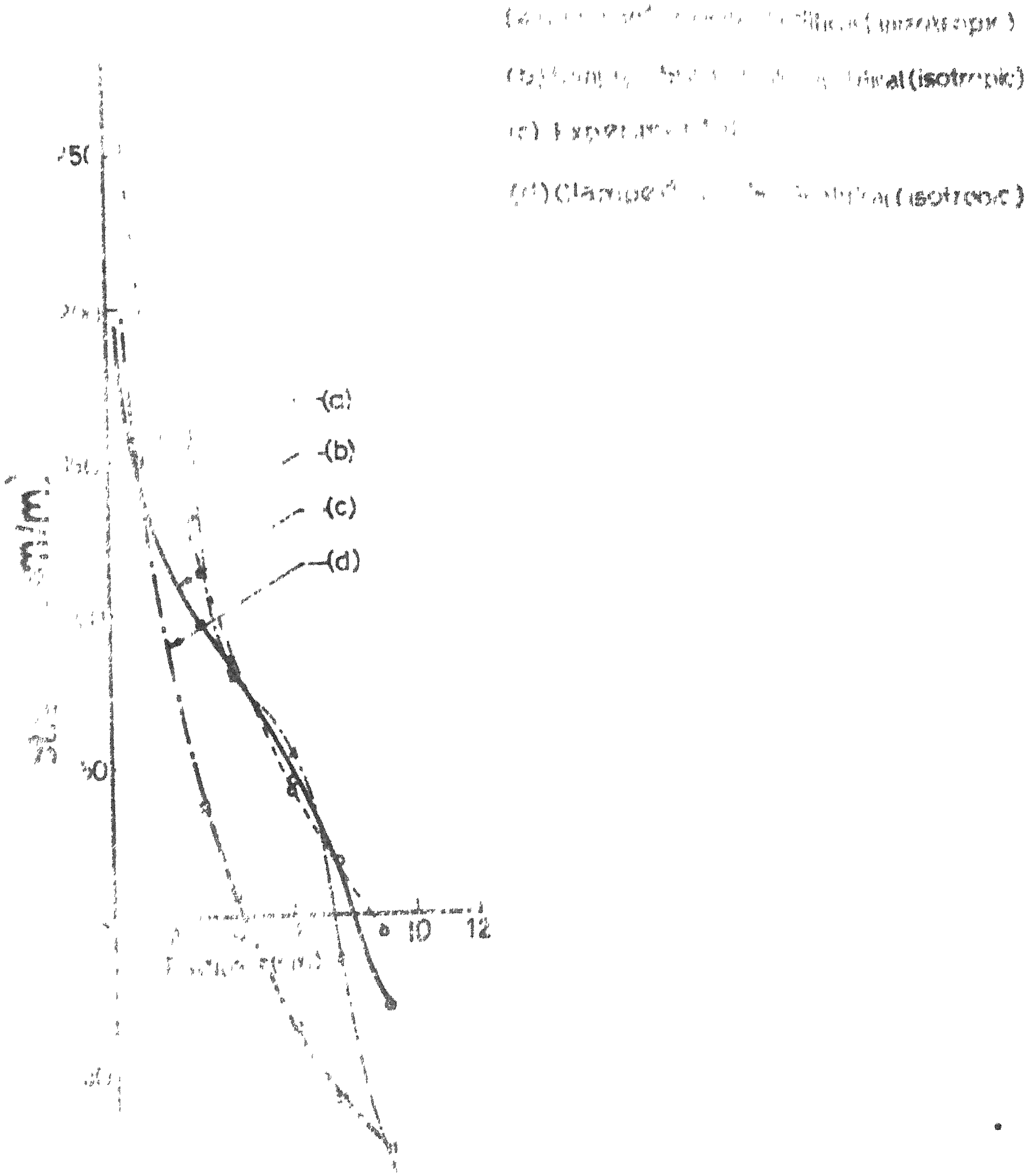


Fig.[illegible] Sectional Element Under General Loading



3.2 Variation of

CHAPTER-4

MODEL FABRICATION AND EXPERIMENTAL SET-UP

4.1 Fabrication of the Model

As the cylindrically reinforced lamina is not available commercially, it has become necessary to fabricate the same in the laboratory. An ingeneous method of casting the cylindrically reinforced glass epoxy composite lamina was developed by Sinha et al [21]. In the present work modifications were made in fibre winding set up for fabrication of composite material discs. The set up designed and fabricated is shown in Fig. 4.1. This set up was used for making cylindrically reinforced glass epoxy composite discs of various thicknesses.

The matrix used was CY-230 epoxy resin cured with 9% HY-951 hardner, whereas the reinforcement used was E glass fibre in roving form. E glass fibre were pulled from the fibre riel successively through a roller and pulley grove dipped in liquid matrix and a closed inclined channel containing the liquid matrix flowing under gravity in the direction opposite to its pull. It was then wound on a very thin stainless steel pin fixed

between two perspex discs which were given sufficient rigidity by supporting aluminium discs.

On completion of the winding, the lamina was pressed in a fixture as shown in Fig. 4.2. The lamina was kept between two mylar sheets and appropriate spacers were used as per the required thickness before tightening the fixture.

4.2 Experimental Set Up

A disc of 24 cm diameter and 6.8 mm thickness, which was fabricated by the fibre winding technique developed in the laboratory was used. The matrix material used for fabricating the disc was CY-230 epoxy resin cured with 9% of HY-951 hardner and the reinforcement was provided by E-glass fibre in roving form.

The disc was machined to the size on a routing machine. Along one of the diameter of the disc twentyone strain gauges in the form of seven rossets (SR-4) were fixed at different radial locations as shown in Fig. 4.3.

Prior to taking readings, the gauges were initially cycled alternately by loading and unloading.

The disc was fixed in the simply supported mode in a mild steel fixture consisting of two mild steel plates as shown in Fig. 4.4.

The eccentric point loading was carried out in cantilever type of loading frame which was designed for this specific experiment. The strain developed was directly noted through a Budd type strain indicator. The disc was rotated in succession by an angle of fifteen degrees. The process was repeated for all the twentyone strain gauges.

The composite material requires the determination of four material properties viz. E_θ , E_r , $G_{\theta r}$ and $\nu_{\theta r}$ to obtain the stress developed when the plate is subjected to eccentric loading. These material properties as given in Table 4.1 are taken from the work reported by Sinha [21] for CY-230 with 9% HY-951 epoxy resin with 40% fibre volume fraction composite material.

Table 4.1

Particular	E_θ (Kg/cm^2) X 10^5	E_r (Kg/cm^2) X 10^5	$G_{\theta r}$(Kg/cm^2) X 10^5	$\nu_{\theta r}$
Experimental	2.541	0.494	0.160	0.197
Theoretical	3.180	0.133	0.161	0.323

The coefficients of the generalized Hooke's law were obtained from the material constants as -

$$a_{11} = \frac{1}{E_\theta} = 3.9355 \times 10^{-6} \ \mathrm{cm}^2/\mathrm{Kg}$$

$$a_{22} = \frac{1}{E_r} = 20.243 \times 10^{-6} \ \mathrm{cm}^2/\mathrm{Kg} \qquad (4.1)$$

$$a_{12} = -\frac{\nu_{\theta r}}{E_\theta} = 0.7753 \times 10^{-6} \ \mathrm{cm}^2/\mathrm{Kg}.$$

The strain gauge rossetes were fixed at different radial locations as shown in Fig. 4.3. The Budd type strain indicator was used to record the strains developed at these points when load was applied at different locations.

The stresses σ_r and σ_θ at these points are given by

$$\sigma_r = \frac{a_{11}\ \sigma_r - a_{12}\,\sigma_\theta}{a_{11}\ a_{22} - a_{12}^2}$$

$$\sigma_\theta = \frac{a_{22}\ \sigma_\theta - a_{12}\,\sigma_r}{a_{11}\ a_{22} - a_{12}^2} \qquad (4.2)$$

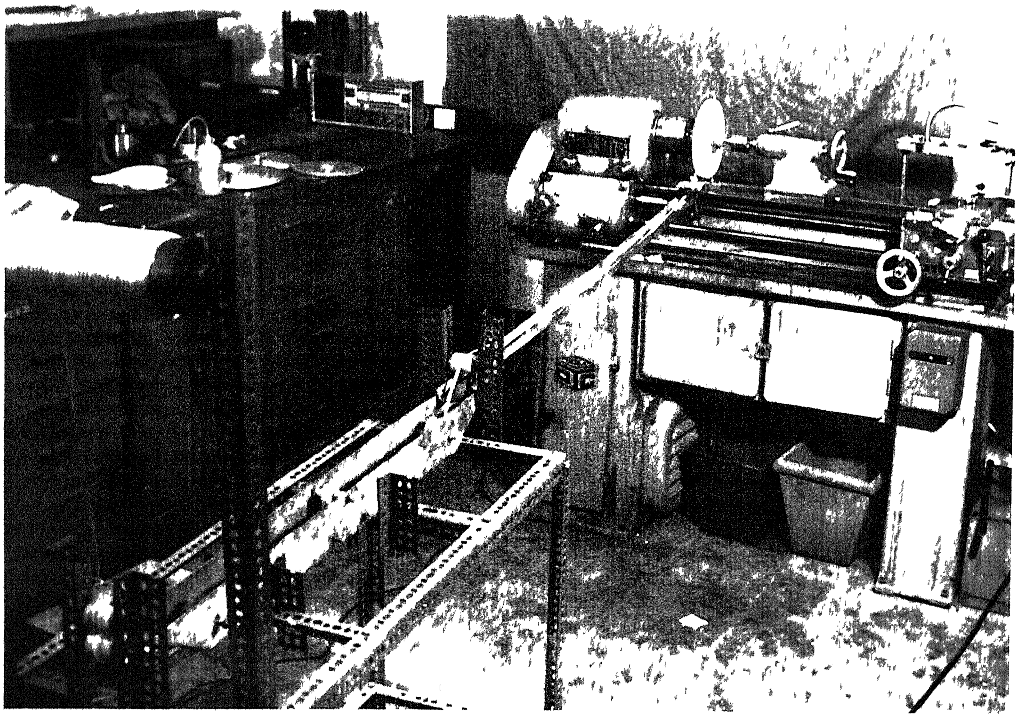


Fig. 4.1 : Fabrication Set Up for the Preparation of Cylindrically Anisotropic Discs.

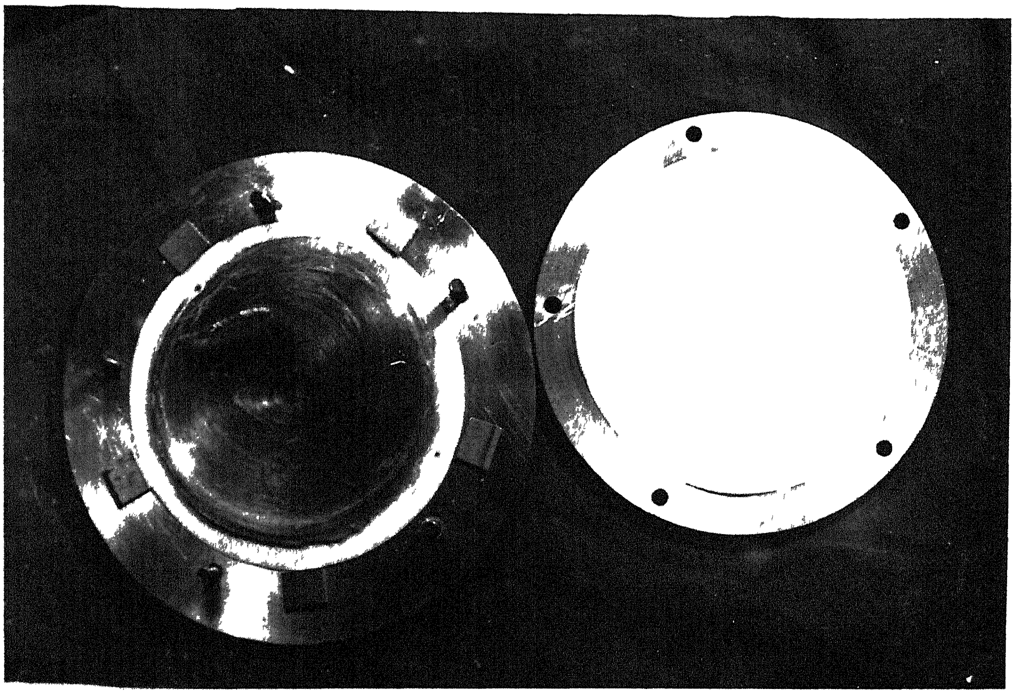


Fig. 4.2 : Details of Fixture Used for Curing the Cylindrically Anisotropic Lamina.

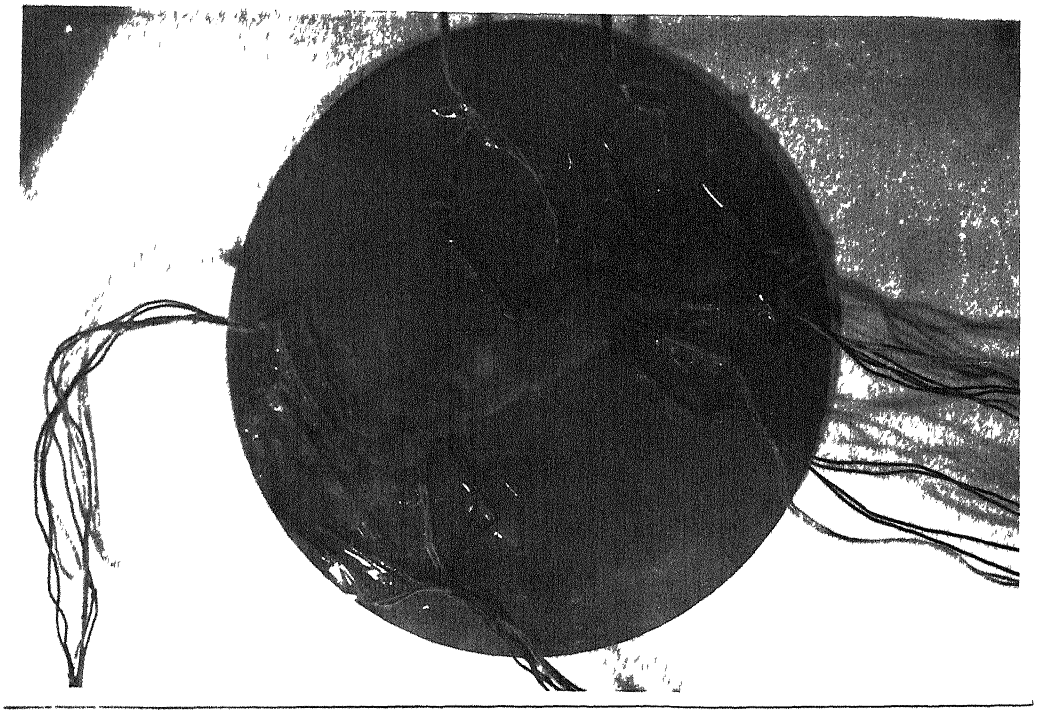


Fig. 4.3 : Experimental Disc with Strain Gauges Fixed at Different Radial Positions.

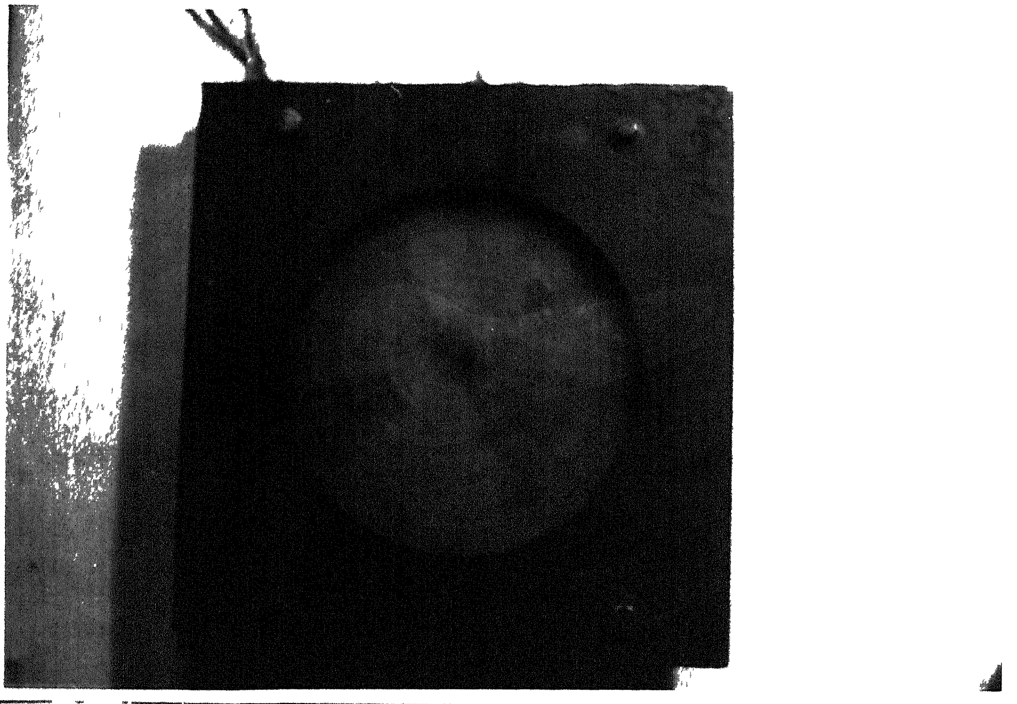


Fig. 4.4 : Loading Fixture with Anisotropic Disc.

CHAPTER-5

RESULTS AND DISCUSSION

Figure 5.1 gives the radial variation of σ_r and σ_θ when concentrated load was applied at the centre of the plate. The theoretical stress distribution in an isotropic plate subjected to central load as given in Fig. 5.2 is reported by Timoshenko et al [23]. while comparing Fig. 5.1 and Fig. 5.2, it is observed that, although the pattern of stress variation in an anisotropic plate is similar to that of an isotropic plate, however, near the centre, the magnitude of σ_θ is much larger compared to the magnitude of σ_r in anisotropic plate, whereas the magnitudes of σ_r and σ_θ are equal for an isotropic case. This discrepancy is because of the anisotropic nature of the plate, where E_θ is not equal to E_r. As the direction of fibre orientation is circumferential, the circumferential modulus of elasticity E_θ is much larger compared to radial modulus of elasticity E_r.

Figure 5.3 is a reproduction of work reported by Timoshenko et al [23], giving the pattern of circum ferential variation of Bending Moment and hence bending

[illegible] eccentric load is applied at a radius of [illegible] $R = \frac{3a}{5}$ on a simply supported isotropic [illegible].

Figure 5.4 to Fig. 5.6 and Fig. 5.7 to Fig. 5.10 show the circumferential variation of σ_r and σ_θ respectively, at radial positions of r = 3 cm, 6 cm and 9 cm when eccentric load is applied at radial locations of R = 1 cm, 3 cm, 5 cm, 7 cm and 9 cm. The pattern of stress distribution obtained is similar to the one reported by Timoshenko for isotropic plate as shown in Fig. 5.3.

It has also been observed that the angular dependence of σ_r is much more strong compared to that of σ_θ. At a rotation of 180°, the magnitude of σ_r either reaches zero or becomes negative, whereas the magnitude of σ_θ always remains positive. The reason for weaker dependence of σ_θ with respect to θ is the cylindrical orientation of fibre.

Figure 5.11 to Fig. 5.14 and Fig. 5.15 to Fig. 5.18 show the nature of variation of σ_r and σ_θ respectively, along the radius when eccentric load is applied at different angular positions of 0°, 30°, 60° and 90° at five different radial positions of R = 1 cm, 3 cm, 5 cm, 7 cm and 9 cm.

As the singularity occurs at the point where concentrated load was applied, the stress developed at those points was infinite as shown in Fig. 5.11. The maximum stress also occurred at the radial positions of 1 cm, 3 cm, 5 cm, 7 cm and 9 cm, where the concentrated load was applied successively. However the radial dependence of σ_Θ is not very strong as compared to the radial dependence of σ_r. It was also observed that the magnitude of maximum stress developed reduced successively with angular rotation from $\Theta = 0°$ to $\Theta = 90°$.

In an isotropic body with pin holes at the centre, stress concentration was found to be significant near the pin holes. However, it was observed experimentally that no such stress concentration takes place in a cylindrically anisotropic plate where a pin hole at the centre is unavoidable due to model fabrication constraints.

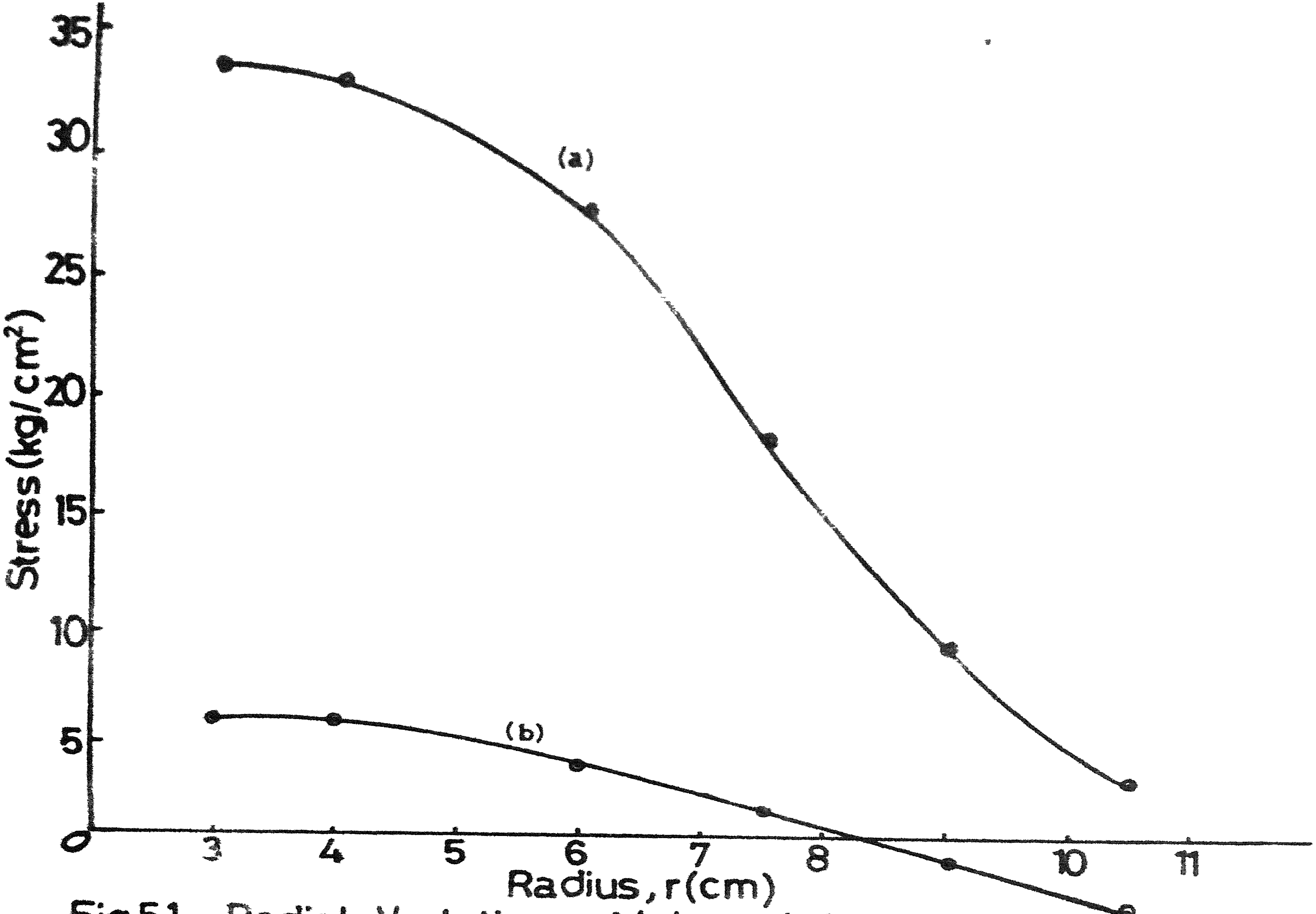


Fig.5.1 Radial Variation of (a) σ_θ (b) σ_r When Load Applied at the Centre of the Plate

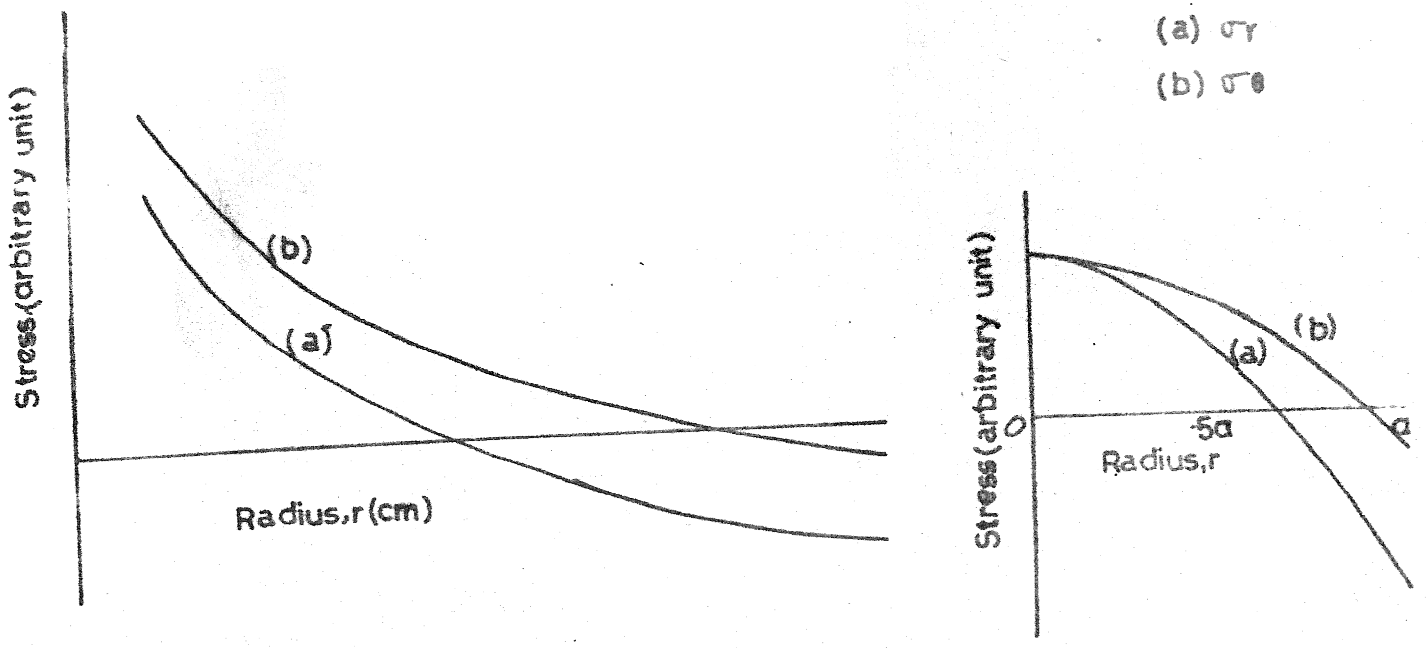


Fig.5.2 Theoretical Variation of Stress When Isotropic Plate Fixed in Simply Supported Mode Subjected to (a) Central Load (b) Uniformly Distributed Load

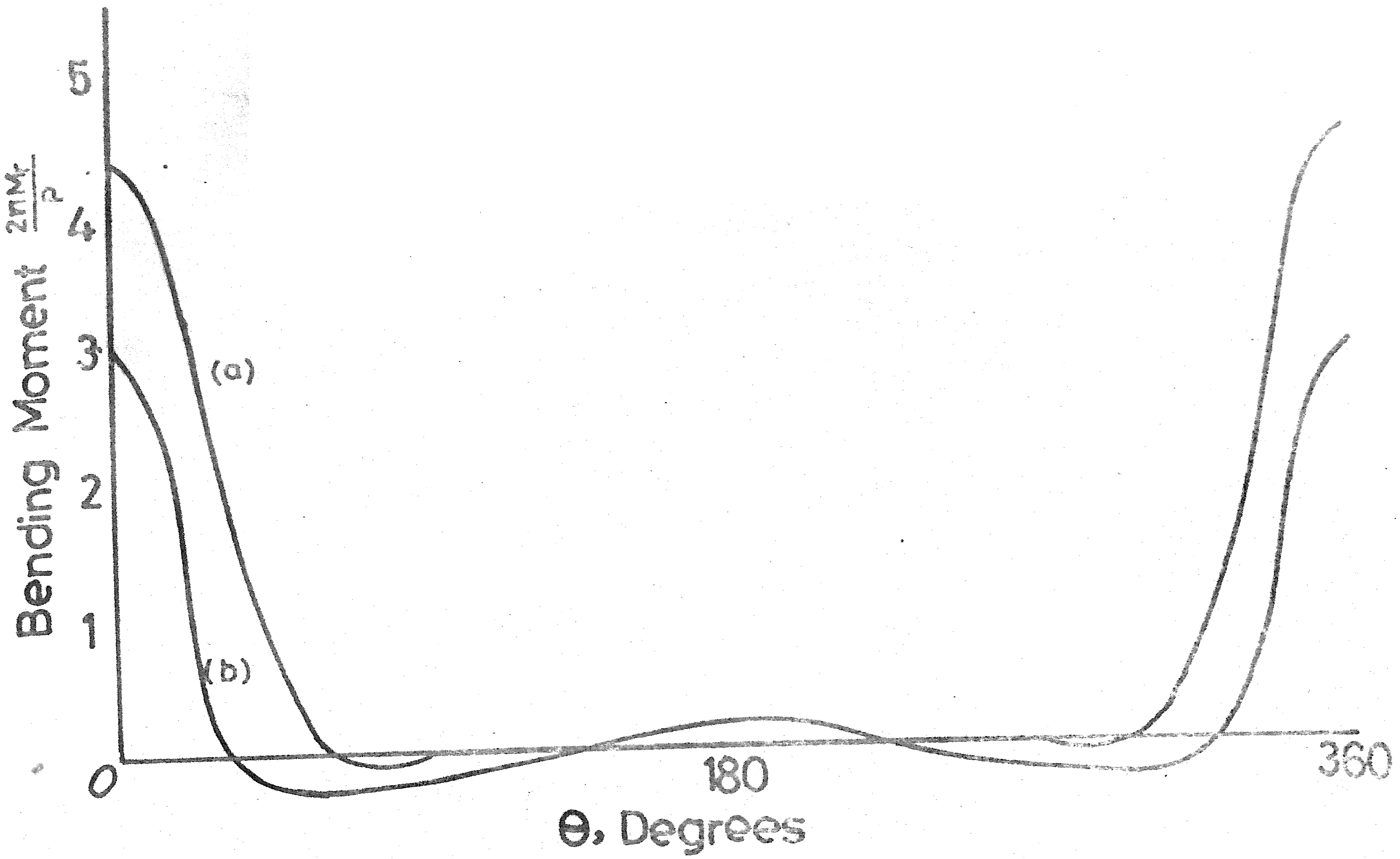
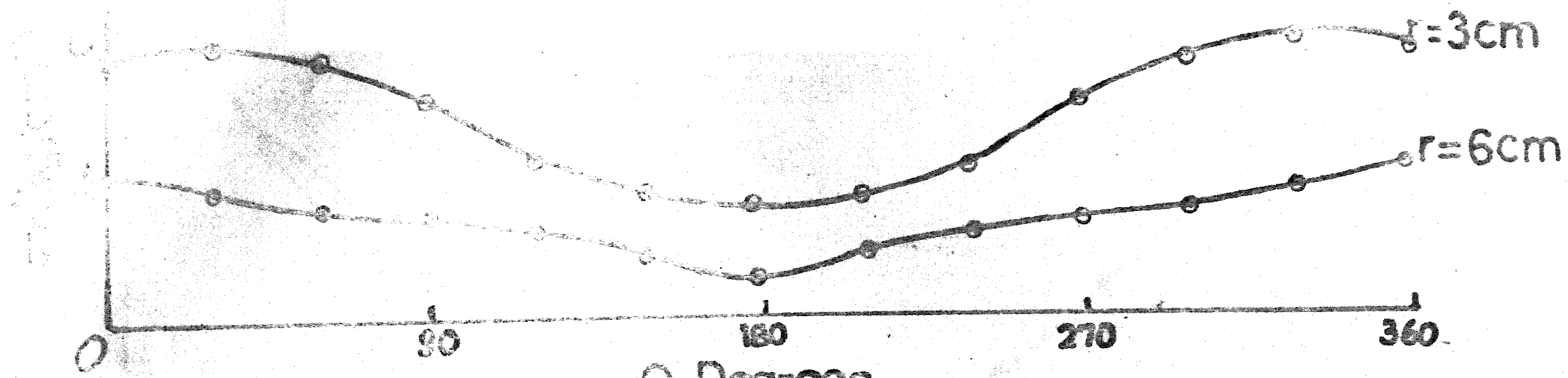
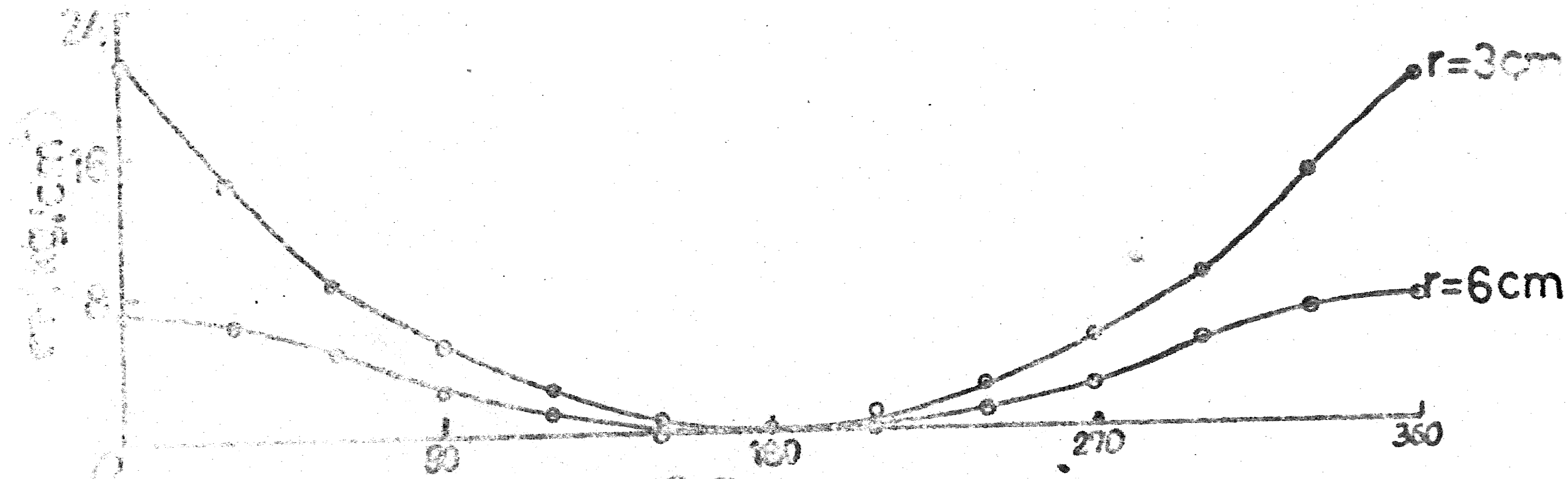


Fig.5.3 Angular Variation of Bending Moment Mr When Eccentric Load Applied at (a) $R=\frac{2a}{3}$ (b) $R=\frac{5a}{6}$ on an Isotropic Plate



(a)



(b)

Fig.5.4 Circumferential Variation of σ_r When Load Applied at (a) R=1cm, (b) R=3cm

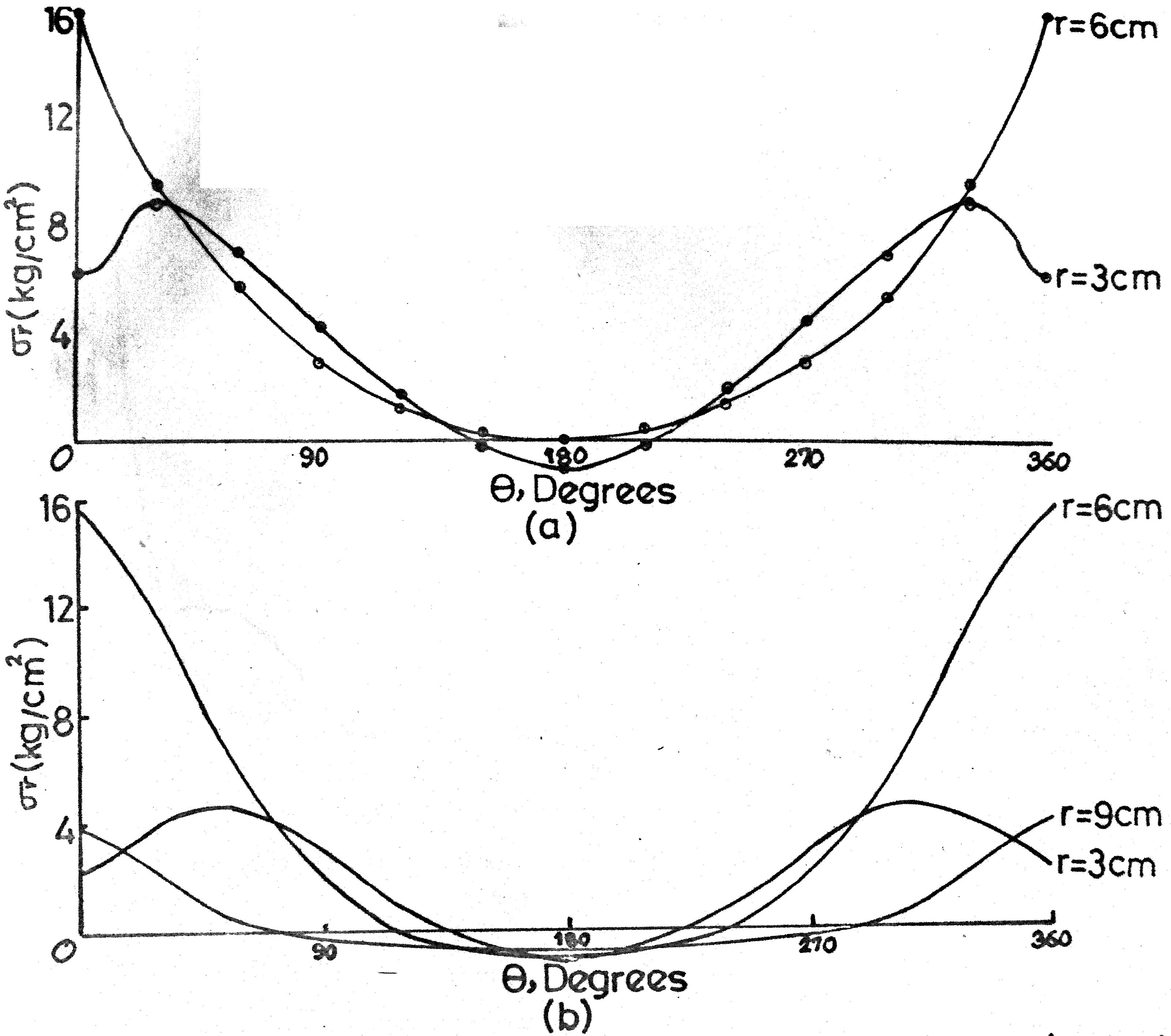


Fig.5.5 Circumferential Variation of σ_r When Load Applied at (a) R=5cm (b) R=7cm

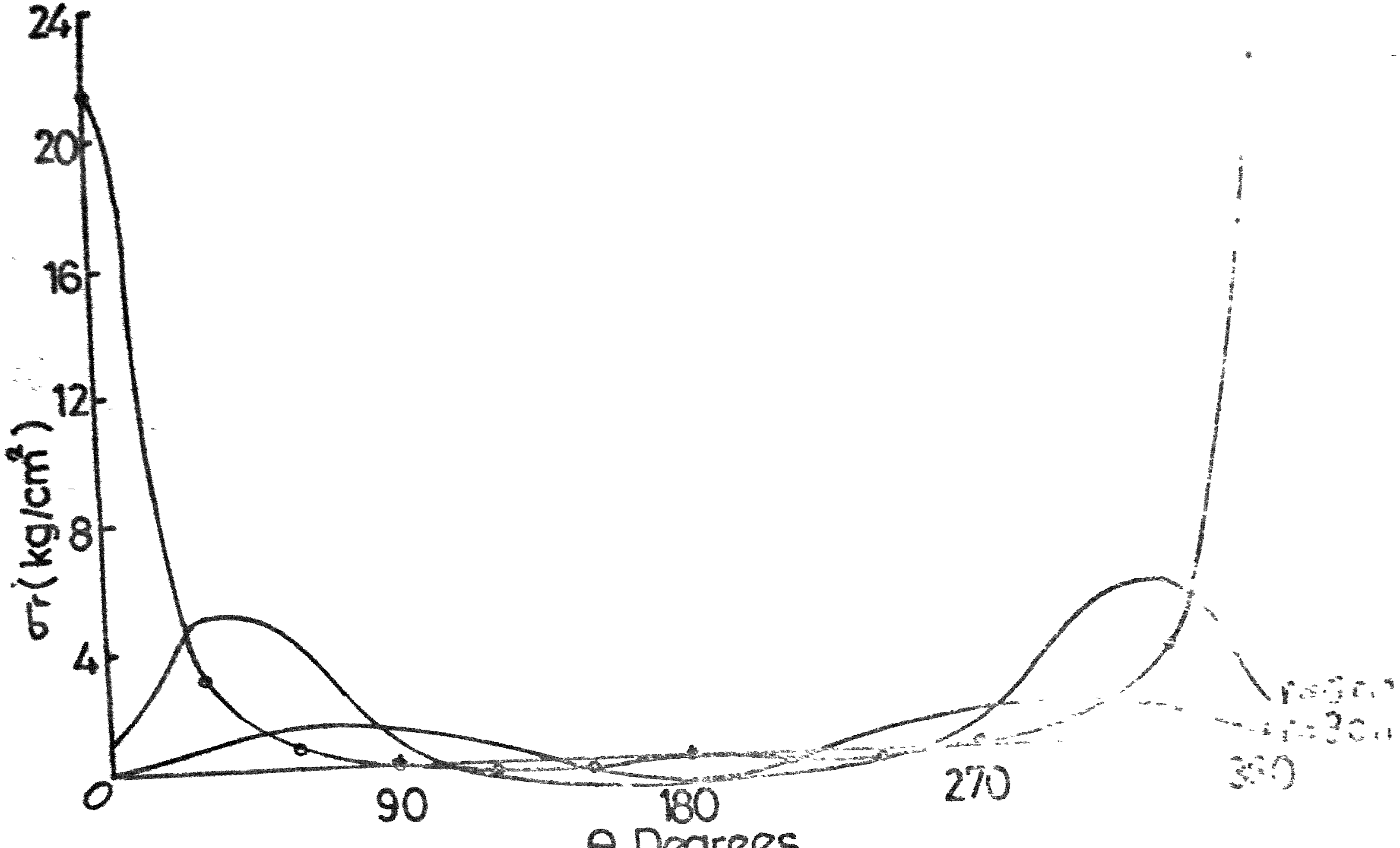


Fig.5.6 Circumferential Variation of σ_r When Loc
at R=9cm

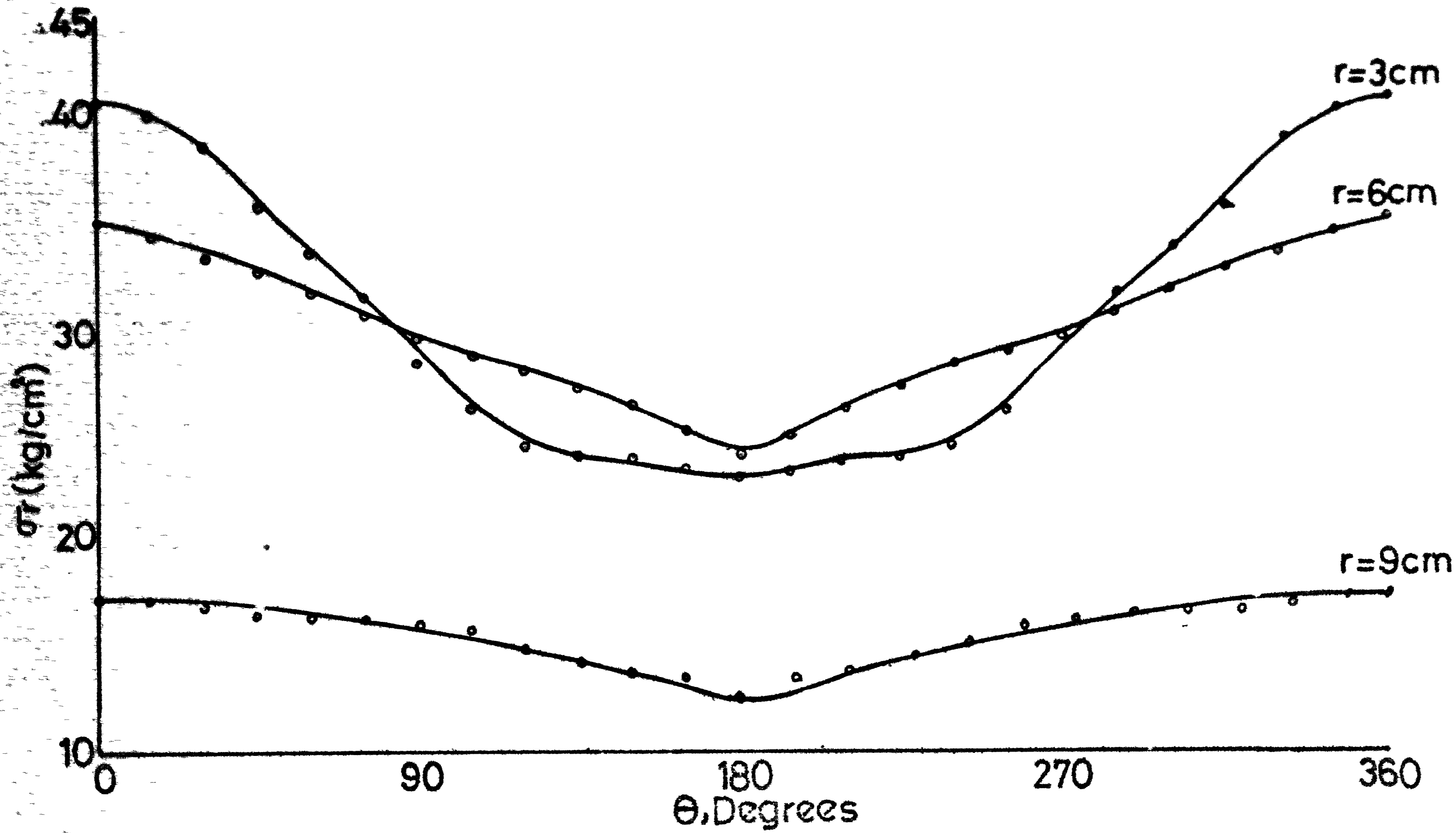


Fig 5.7 Circumferencial Variation of σ_r When Load Applied at R=1cm

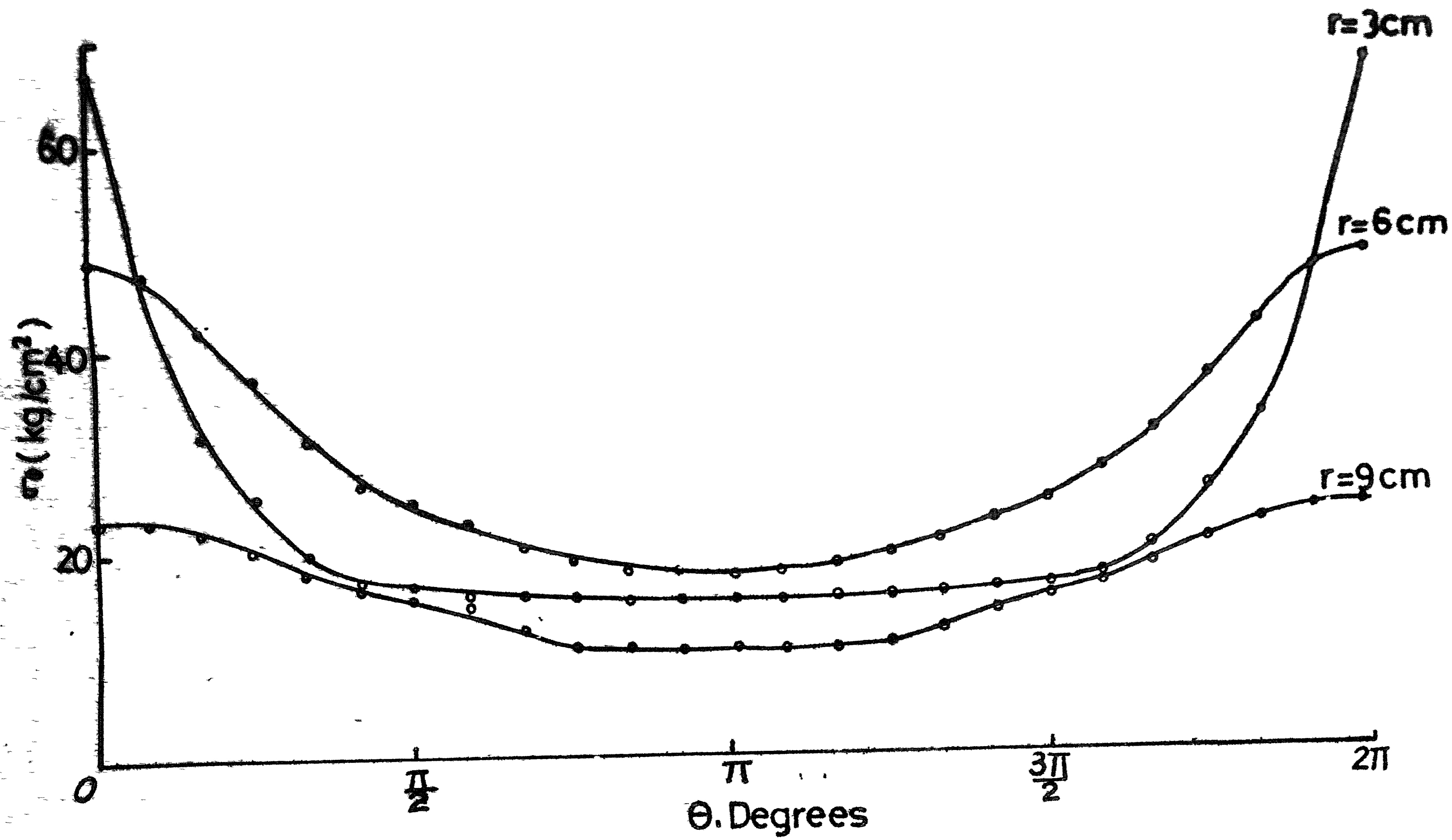


Fig.5.8 Circumferential Variation Of σ_θ When Eccentric Load Applied at R=3 cm

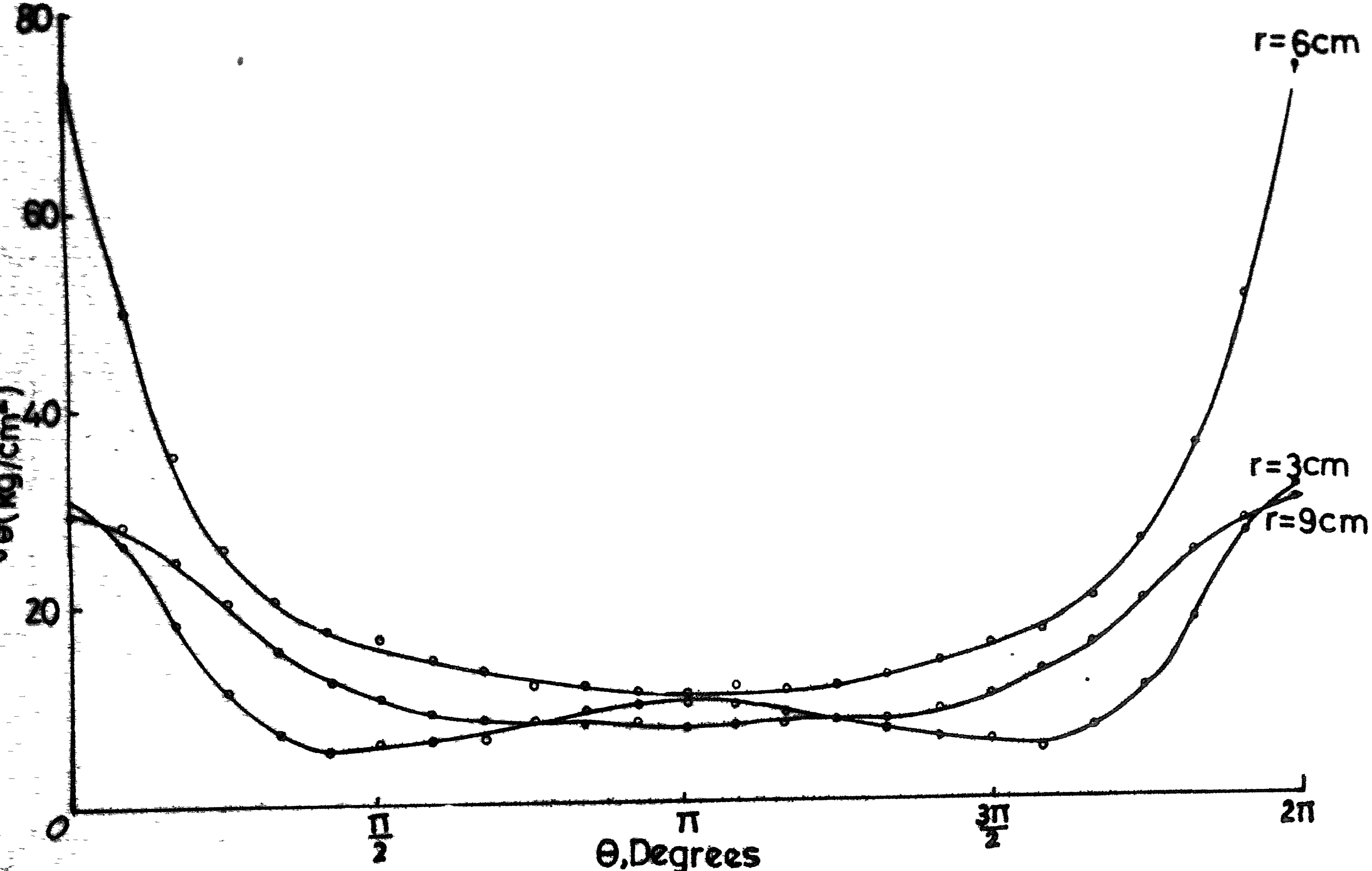


Fig.5.9 Circumferential Variation of σ_θ When Eccentric Load Applied at R=5cm

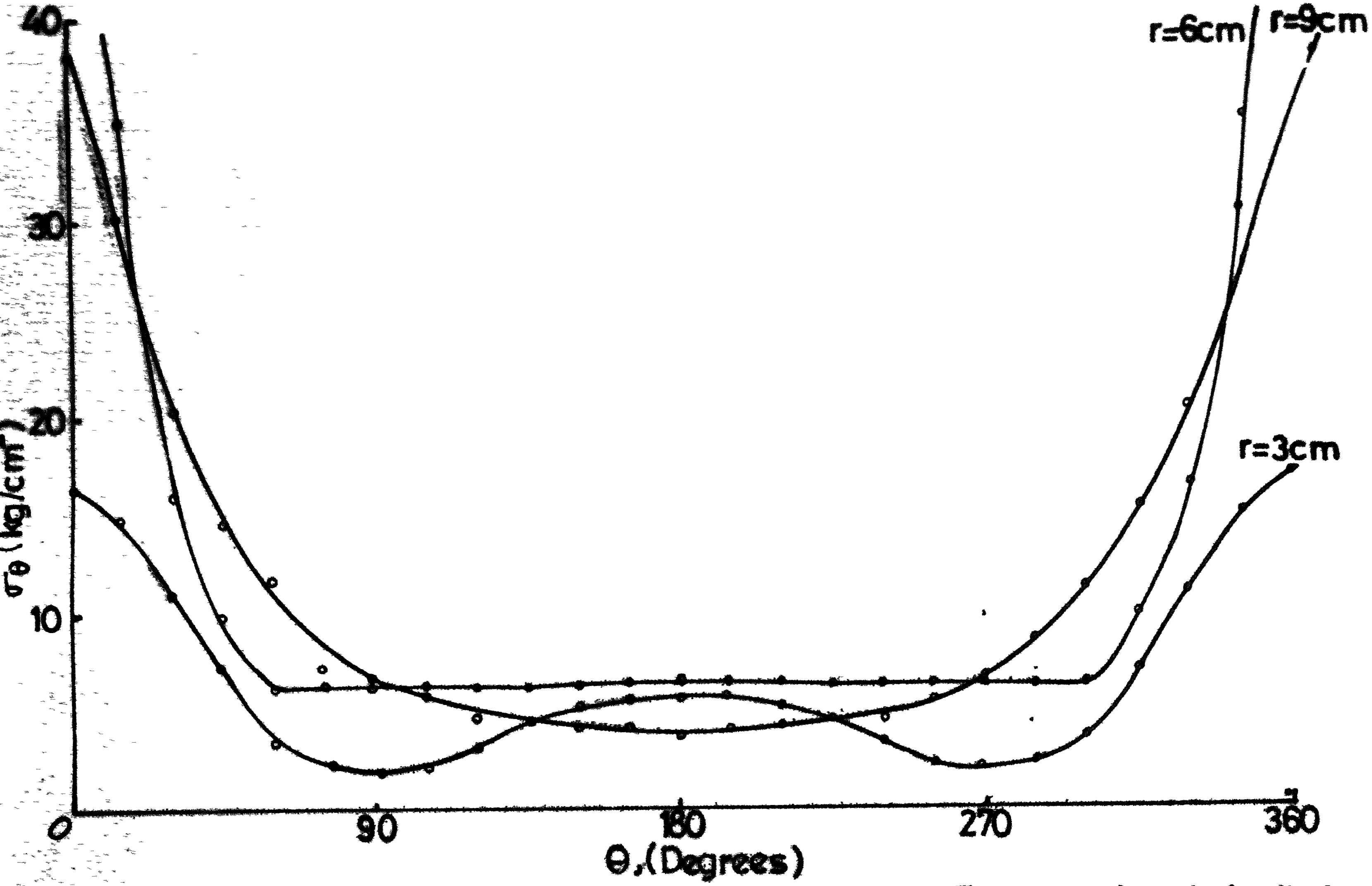


Fig.5.10 Circumferential Variation of σ_θ When Eccentric Load Applied at R=7cm

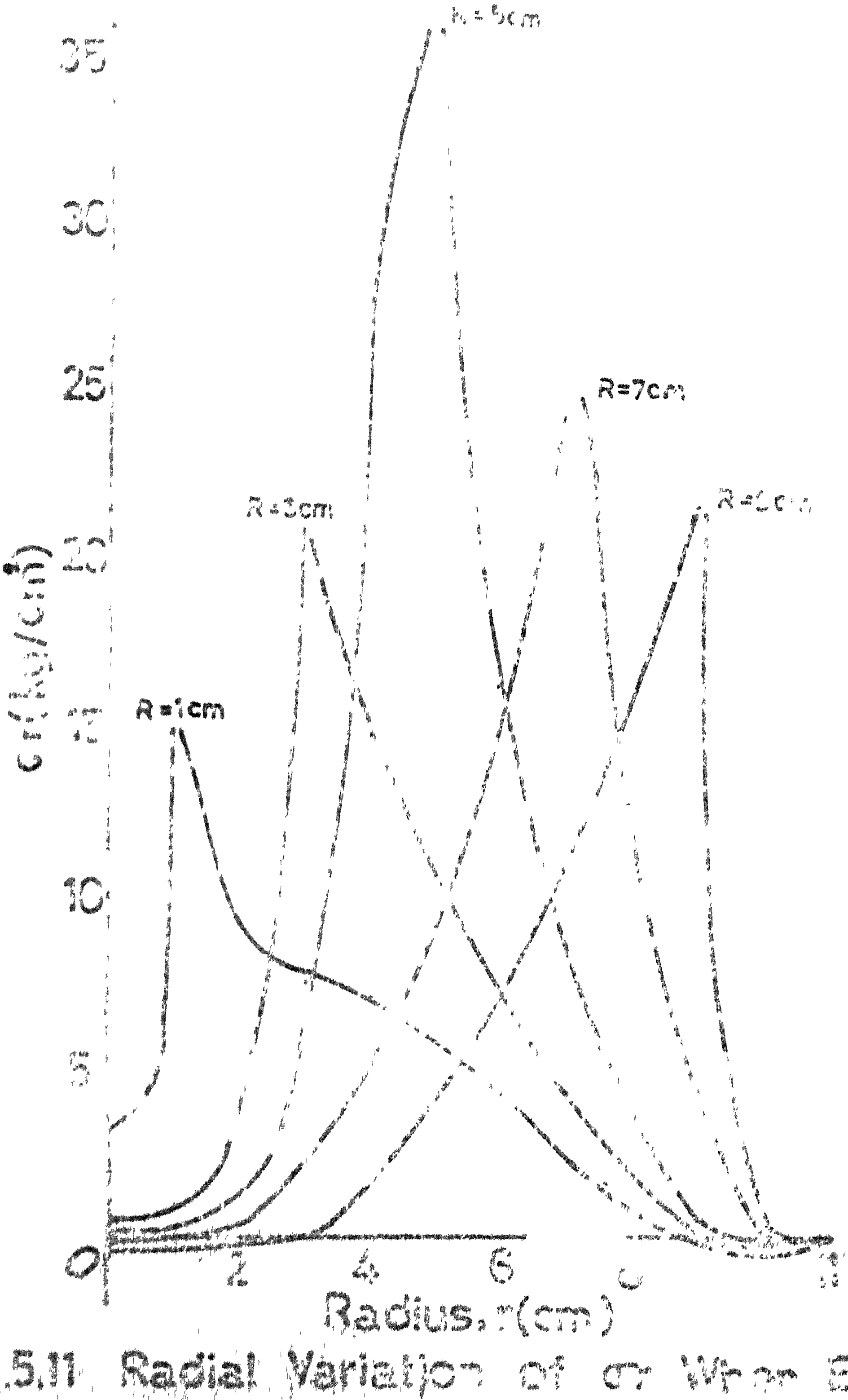


Fig.5.11 Radial Variation of σr When Eccentric Load Applied at θ=0 degrees

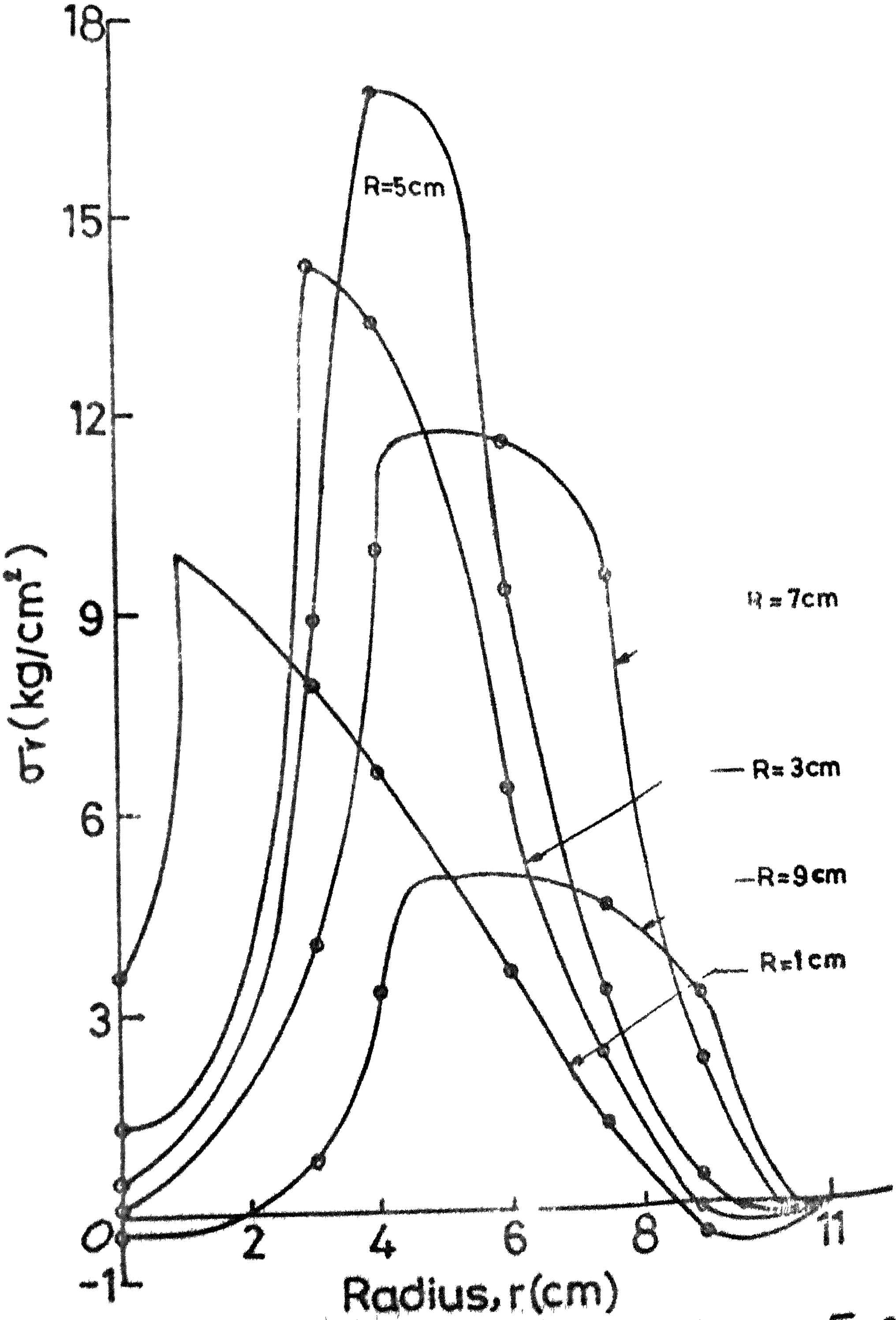


Fig.5.12 Radial Variation of σ_r When Eccentric Load Applied at Θ=30 degrees

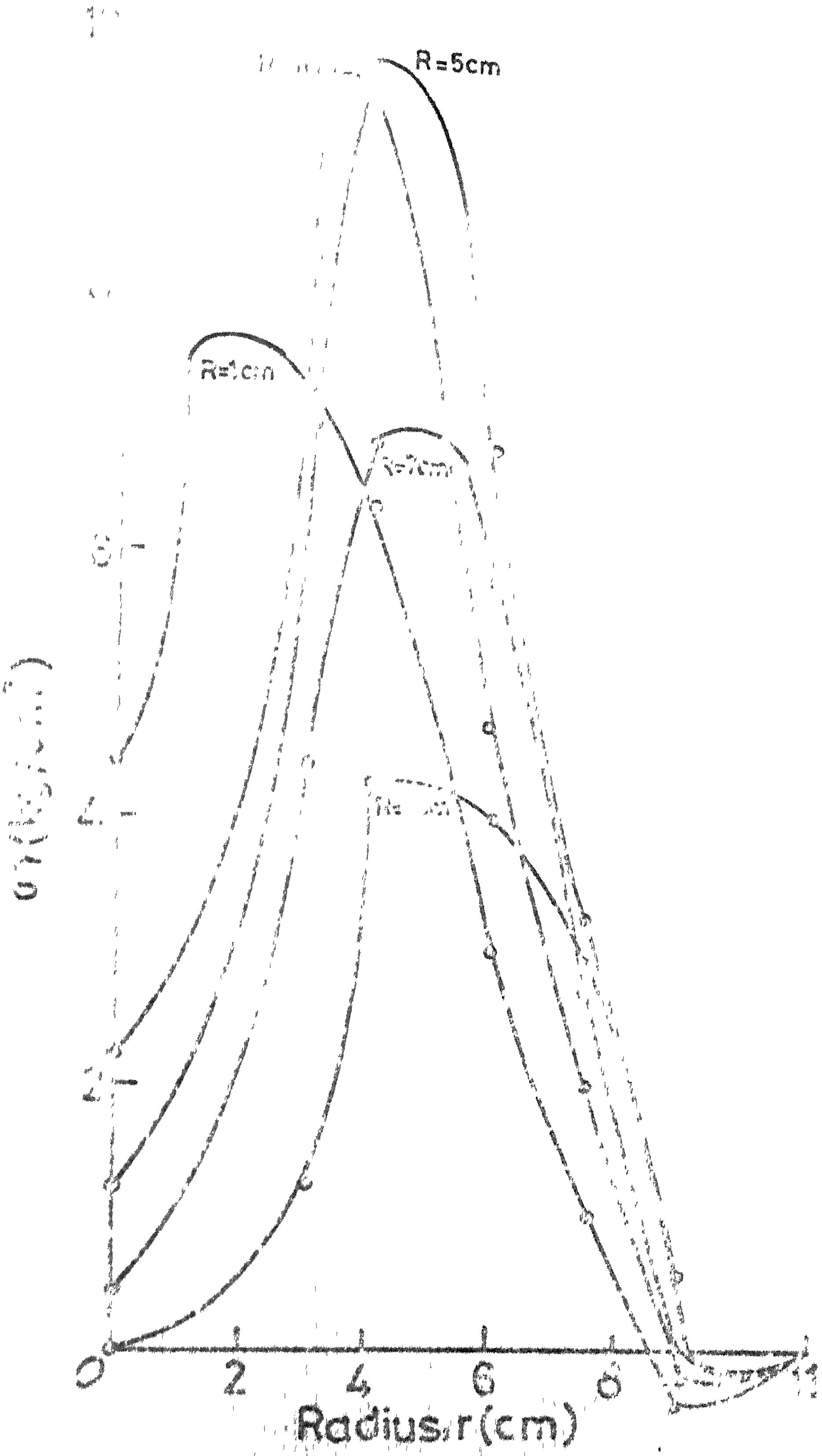


Fig.5.13 Radial Variation of σr When Eccentric Load Applied at θ=60 degrees

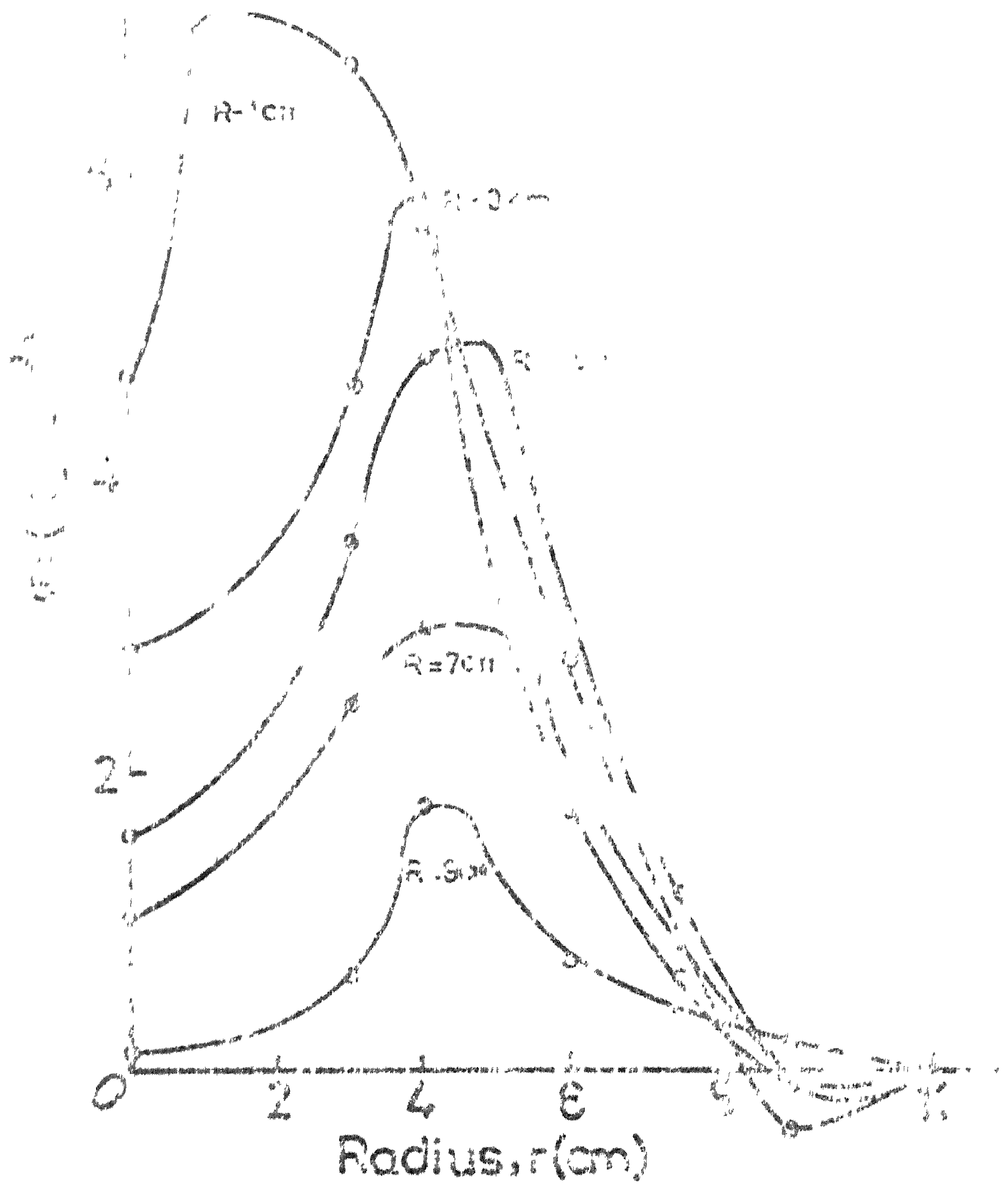


Fig.5.14 Radial Variation of σr When Eccentric Load Applied at Θ=90 Degrees

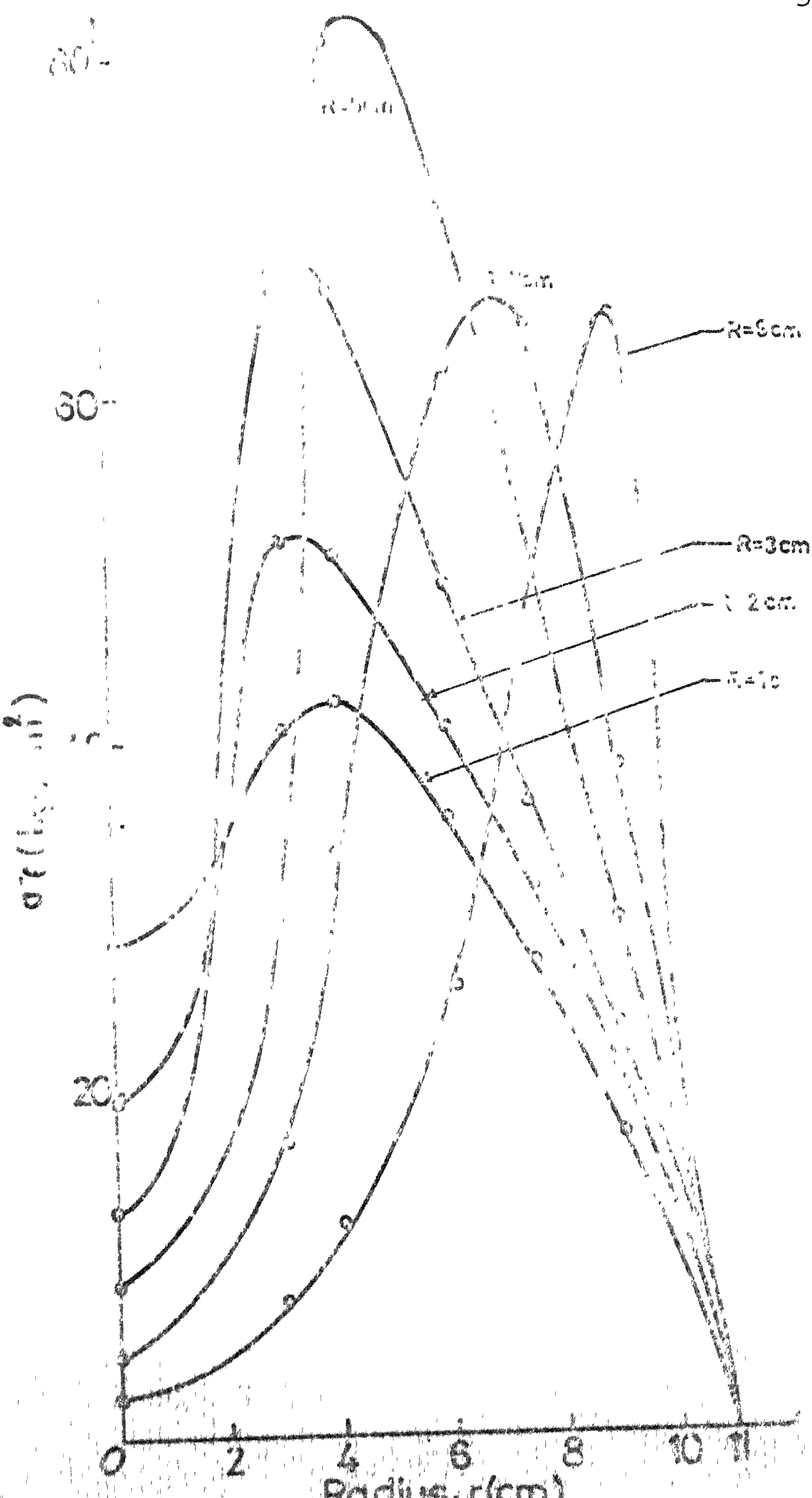


Fig.5.15 Radial Variation of σ_θ When Eccentric

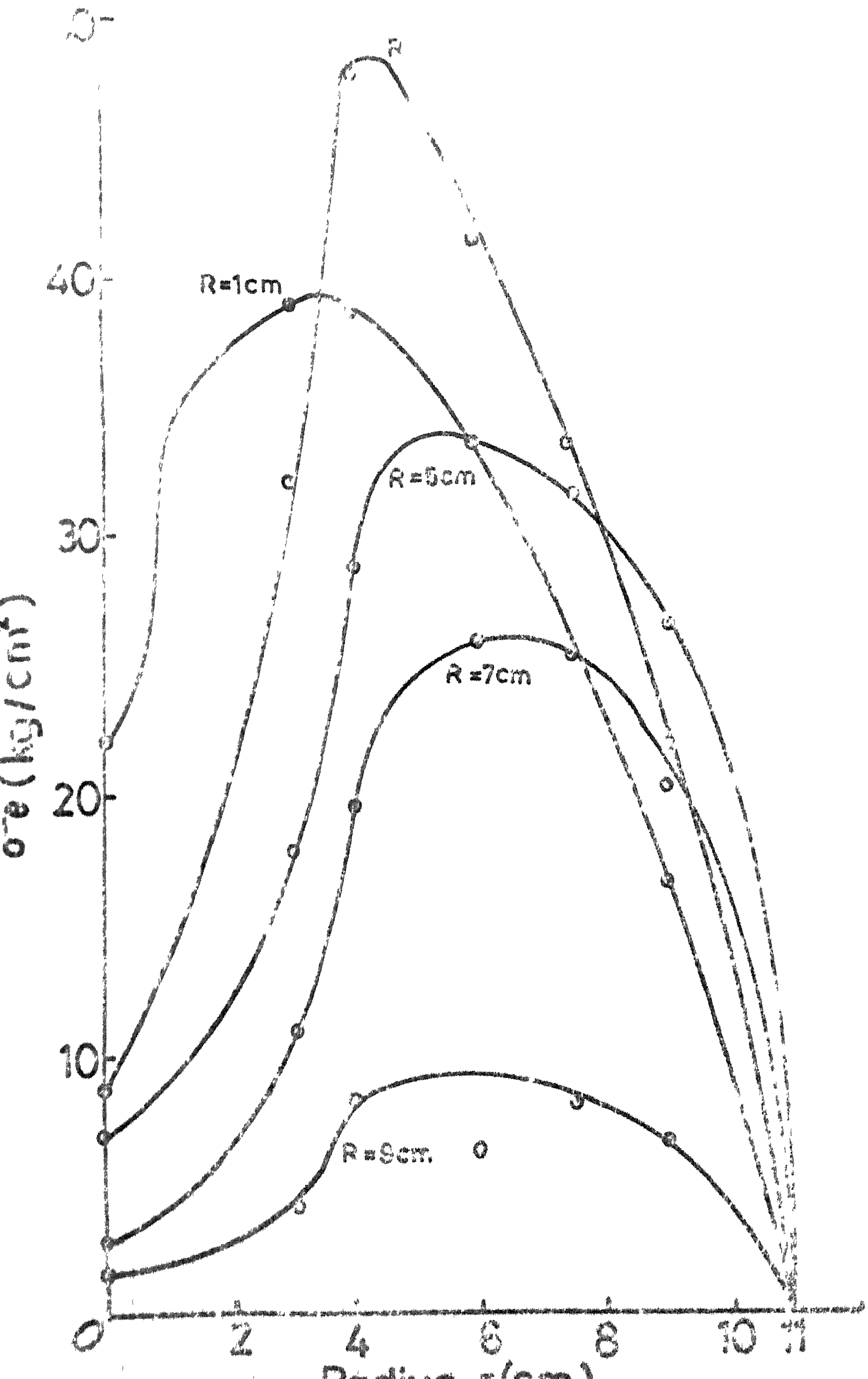


Fig 5.16 Radial Variation of σθ When Eccentric Load Applied at θ=30 Degrees

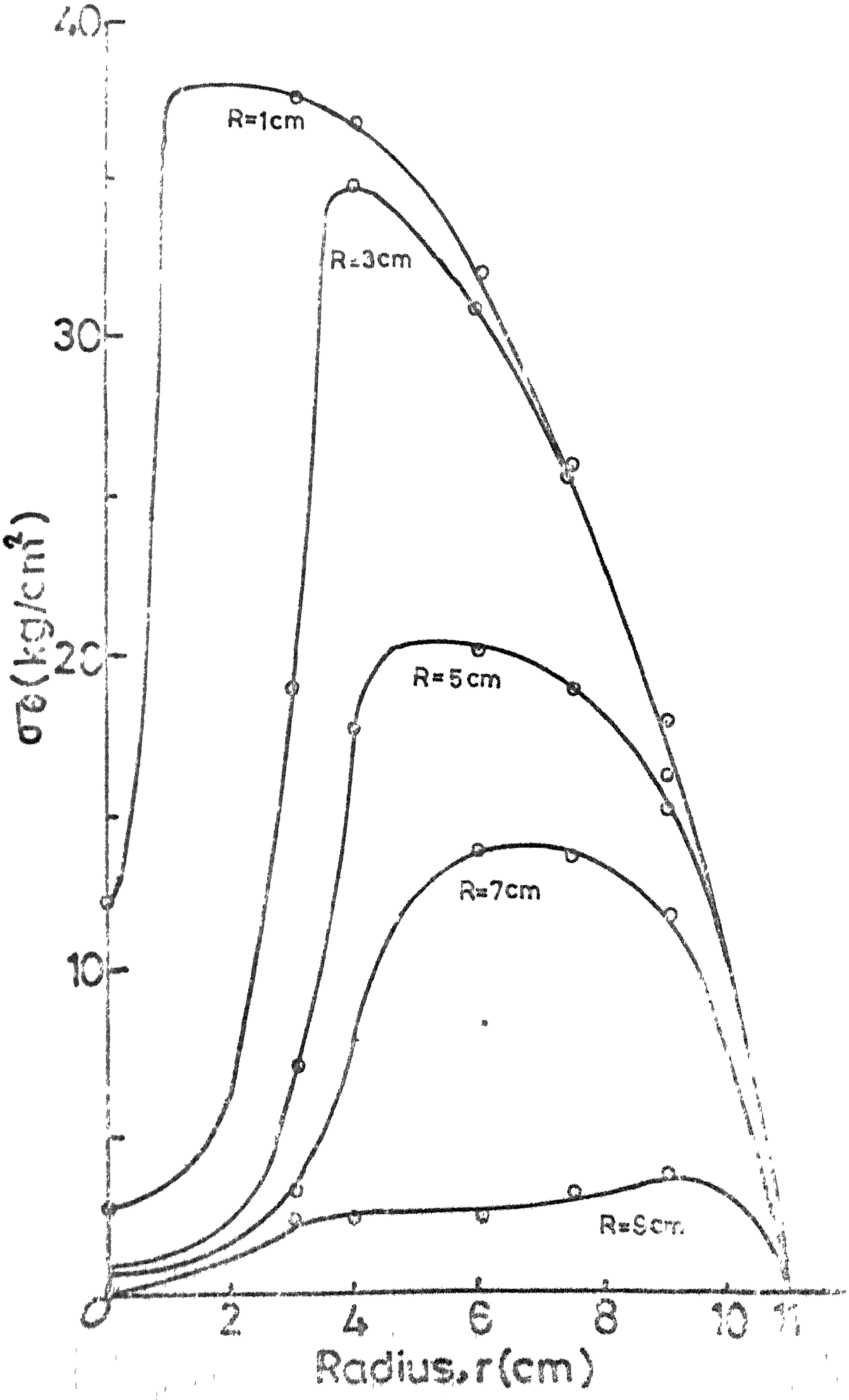


Fig.5.1 Radial Variation of σ_θ When Eccentric Load Applied at θ = 00 Degrees

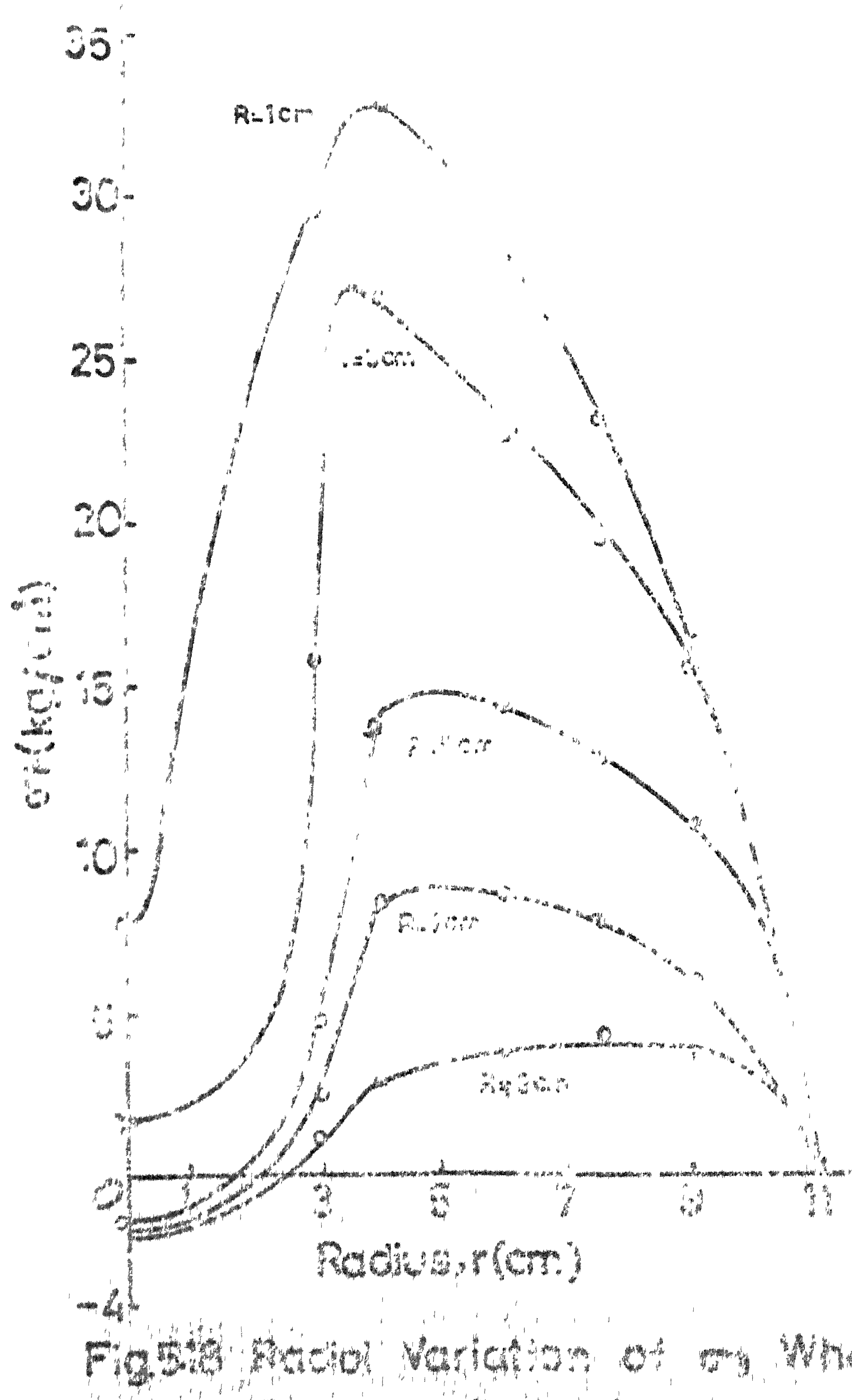


Fig.5.18 Radial Variation of σ_θ When Eccentric Load Applied at θ=90°

CHAPTER-6

CONCLUSION

6.1 In the present work an attempt was made to study the problem of plane deformation of an anisotropic body. The differential equation for stress potential function is obtained for the plane deformation and a procedure is also outlined to obtain the stress function for various other modes.

The major contribution of the present work are as follows:

a) A laboratory method for the fabrication of cylindrically composite disc has been developed.

b) The fabricated models have been tested for the behaviour under eccentric loading.

c) Using the strain gauge technique, stress developed in an anisotropic body subjected to eccentric loading is mapped.

d) No stress concentration is observed near the pin hole which is unavoidable because of the technique used for model fabrication.

[illegible]. A finite element solution to the given differential equation will be useful for comparing with the results experimentally obtained in this work.

ii) An experimental and theoretical study of the disc subjected to dynamic loading should also be done.

iii) The problem of stress concentration near the central hole of various diameters in cylindrically anisotropic plates should be investigated. For isotropic material a high stress concentration near pin hole at the centre of the disc exists, whereas, the stress behaviour of a pin hole near the centre of an anisotropic disc has been found to be normal. This needs to be further probed.

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